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RESEARCH ARTICLE

Bargaining with subjective mixtures

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Abstract This paper reconsiders the Bargaining Problem of Nash (Econometrica 28:155–162, 1950). I develop a new approach, *Conditional Bargaining Problems*, as a framework for measuring cardinal utility. A Conditional Bargaining Problem is the conjoint extension of a Bargaining Problem, conditional on the fact that the individuals have agreed on a "measurement event". Within this context, *Subjective Mixture* methods are especially powerful. These techniques are used to characterise versions of the Nash and the Kalai–Smorodinsky solutions. This approach identifies solutions based only on the individuals' tastes for the outcomes. It is therefore possible to do Bargaining theory in almost complete generality. The results apply to *Biseparable* preferences, so are valid for almost all non-expected utility models currently used in economics.

Keywords Bargaining · Utility · Subjective mixtures · Biseparable preferences

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1 Introduction

Situations where people could, by reaching some agreement, enjoy mutual benefits are pervasive in economic life. Typically, in such situations, there is more than one

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agreement that could be chosen. The question of how an agreement is reached is known as the *Bargaining Problem*. Since Nash (1950), there has been extensive research on axiomatic approaches to the Bargaining Problem. A recent survey of this literature is Thomson (2009).

Following Nash (1950), a two-person Bargaining Problem is typically modelled as a set of utility pairs. Each utility pair assigns a utility value to each individual. One utility pair is designated the *disagreement point*; the utilities each receive should they fail to reach agreement. A *Bargaining Solution* is a rule that uses the data of any Bargaining Problem to select one of the possible utility pairs.

The set of possible utility pairs, associated with a Bargaining Problem, is frequently assumed to be convex. There are several plausible justifications for the convexity assumption. Nash's original interpretation is as follows: assume that each person has preferences represented by expected utility. The set of feasible utility pairs is the image of some underlying set of alternatives. Suppose that lotteries over these alternatives are feasible, then, if two alternatives are graphed to two utility pairs, the lotteries over these alternatives are graphed to a line connecting these utility pairs. It is this interpretation that will be discussed. The use of lotteries is also assumed by many papers addressing the preference foundations of Bargaining Theory (Rubinstein et al. 1992; Grant and Kajii 1995; Safra and Zilcha 1993; Hanany and Safra 2000; de Clippel 2009).

The use of lotteries serves an important purpose in the modelling of the Bargaining Problem. It allows us to *measure utilities*. That is, since the underlying set of alternatives has a *Mixture Space* structure (Herstein and Milnor 1953), it can be proved conclusively that Expected Utility representations exist and, further, that the utility functions for outcomes will be cardinally unique. There are, however, cases where we may not be able to demonstrate the existence of cardinal utility.

Consider the following example: Two professors are given the task of making a "superstar" appointment. They are asked to agree on and hire the most important academic. There are no other concerns. Each professor considers the pool of available academics. Each professor can order the set of academics, in terms of importance, although their orders do not agree. They approach an arbitrator who is trained in the tools of utility measurement. The arbitrator takes their 'importance orders' and constructs numerical representations. These representations of the importance orders are, essentially, ordinal utility functions. The two sets of importance rankings, however, lack a suitable notion of *concatenation*. That is, we are unable to say that a professor considers one academic to be "twice as important as another" because we are unsure what "twice as important " means. Unless we consider the mixed, or some other extension of the problem, I believe we can do no better than ordinal utility (Krantz et al. 1971, p. 123). A stronger statement is made by Fleurbaey and Hammond (2004):

> "If X is simply a set of riskless alternatives, then it is impossible to derive cardinally measurable utility functions from individual preferences over this set alone . . . it needs preferences over lotteries, for instance." (p. 1227)

We have a simple problem, "agree on an academic". A sensible solution should reflect the preferences of each individual, and also involve some compromise. A result of Shapley (1969) states, roughly, that it is *impossible* to find a sensible solution using only ordinal information. Defining a sensible solution requires further knowledge about the structure of preferences. Shapley's theorem says that *we really need utilities to be cardinal*, and Herstein and Milnor's theorem reassures us that *the use of lotteries delivers this*.

Probability provides the currency with which a von Neumann–Morgenstern utility is measured. Expected Utility is linear in probabilities. This corresponds to what has become known as *Probabilistic Risk Neutrality* (Wakker 1994). Empirical evidence suggests people's preferences are not typically neutral in this sense (Kahneman and Tversky 1979; Machina 1987). Non-expected utility is the theory of such preferences.

Most Bargaining Solutions are distorted by probabilistic risk attitudes. Köbberling and Peters (2003) demonstrated that, in the context of the Rank-Dependent Utility model (Quiggin 1982), probabilistic risk attitudes affect the solution of Kalai and Smorodinsky (1975) in a predictable way. Other things being equal, you will never be better off if you bargain against someone more probabilistically risk averse.

Consider the "agree on an academic" problem, extended to include lottery agreements. Suppose it is agreed, using the Kalai–Smorodinsky solution, that each professor will receive some chance of appointing the academic they consider most important, then the chance each professor receives will be affected by the probabilistic risk attitudes of the other. The fact that the mixed extension has been used has, in itself, introduced biases that have distorted the solution. Such attitudes, however, have nothing to do with the professors' judgements about the academics' importance. It is my opinion that, where possible, *the solution to the problem should depend only on each individual's preferences over the outcomes*.

To address the problem discussed above, this paper introduces a new approach. Section 3 outlines the concept of a *Conditional Bargaining Problem*. This may be thought of as the 'conjoint extension' of a Bargaining Problem. This approach allows us to apply the modern tools of Conjoint Measurement; previously applied to establish the foundations of many non-Expected Utility models. In particular, the *Subjective Mixture* techniques of Ghirardato et al. (2003) are especially powerful. We outline Subjective Mixtures in Sect. 4. Section 5 gives two applications of the above techniques, providing characterisations of the Nash and the Kalai–Smorodinsky Bargaining Solutions, as applied to Conditional Bargaining Problems. These applications highlight the usefulness of applying Subjective Mixtures in the context of Conditional Bargaining Problems. *The result is a theory of Bargaining that works in almost complete generality*. That is, the results are robust to the majority of non-Expected Utility models used in economics.

2 Nash's Bargaining Problem

In this Sect. 3 outline the two-person bargaining model introduced by Nash (1950). In Nash's formulation, a *Bargaining Problem* is a tuple $\langle U, a\rangle$. The *Feasible Set* is a set utility of pairs, $U \subset \mathbb{R}^2$. A point $(\alpha, \beta) \in U$ assigns a utility value of α for individual 1 and β for individual 2. The *Disagreement Point*, $a = (a_1, a_2) \in U$, is the utilities that each individual receives if they fail to agree on any other candidate from U.

Nash assumes that U is compact and convex. Nash also assumes there is some $\beta \in U$ such that $\beta_1 > a_1$ and $\beta_2 > a_2$. A Bargaining Problem is called compact or convex if U has the associated property. Nash's interpretation of the assumed convexity is as follows: The set of utility pairs is the expected utility image of some set of alternatives and, whenever two alternatives are feasible, any lottery over them is feasible.

A weaker condition than convexity is *Comprehensiveness*. A problem $\langle U, a\rangle$ is a-Comprehensive if $y \geqslant x \geqslant a$ and $y \in U$ jointly imply $x \in U$. The *Comprehensive hull* of a set U with respect to a point a, denoted $comp[U; a]$, is the smallest a-Comprehensive set containing U. We will make use of the following notation:

$$\Phi_1(\alpha; U) := \{\beta \in \mathbb{R} : (\beta, \alpha) \in U\}$$

So $\Phi_1(\alpha; U)$ is the set of utility values that are feasible for individual 1 when 2 gets utility α. $\Phi_2(\alpha; U)$ is defined similarly.

Let $\mathscr{B}$ be some class of Bargaining Problems. A *Bargaining Solution* defined over $\mathscr{B}$ is a function, $S : \mathscr{B} \to \mathbb{R}^2$, that selects, for any Bargaining Problem, a unique and feasible point. Specific solutions can be obtained by restricting S to satisfy certain consistency properties or *axioms*. Section 5 outlines two such Bargaining Solutions: the Nash and the Kalai–Smorodinsky Solutions.

3 Conditional Bargaining Problems

The tuple $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$ is a *Conditional Bargaining Problem*. Here, $A \subseteq X_1 \times X_2$ is the *feasible set of alternatives*. Elements of X_i are x_i, called *outcomes*. So an alternative is a pair (x_1, x_2) where each individual receives an outcome. The *Disagreement Alternative* is $d = (d_1, d_2)$.

It is useful to introduce, briefly, the Savage (1954) framework. There is a set of *states*, $\mathscr{S} = \{\ldots, s, \ldots\}$. The individuals do not know which state will obtain, but that only one will. Subsets of $\mathscr{S}$ are *events*, the set of which is $\mathscr{E} = 2^{\mathscr{S}} = \{\ldots, E, E', \ldots\}$. For ease of exposition, we will assume $\mathscr{E}$ is finite. Acts are functions from states to outcomes $f : \mathscr{S} \to X_i$.

Write $x_i E y_i$ for the act with outcome x_i if $s \in E$ and y_i otherwise. An act $x_i E y_i$ is a *Binary Act*. Preferences $\succcurlyeq_i$ are defined over the set of Binary Acts. All events are *Essential* with respect to each individual's preferences: $x_i \succ_i x_i E y_i \succ_i y_i$ for some $x_i, y_i \in X_i$. Outcomes $x_i \in X_i$ are naturally identified with constant Binary Acts, $x_i E x_i$, and the restriction of preferences to outcomes is also written $\succcurlyeq_i$. I make the simplifying assumption that there are no two distinct outcomes, for either individual, such that $x_i \sim_i x'_i$. This will amount to preferences being antisymmetric. A minor modification of the theory, to dispense with this assumption, will mean solutions are identified only up to their equivalence class (see Rubinstein et al. 1992, p. 1173). All Conditional Bargaining Problems will satisfy the following structural assumption:

Structural Assumption 1 *X_1 and X_2 are compact and connected in some topologies, $\mathscr{T}_1$ and $\mathscr{T}_2$, respectively. A is compact in the product topology $\mathscr{T}_1 \times \mathscr{T}_2$. The disagreement alternative is feasible and there is some $(x_1, x_2) \in A$ with $x_1 \succ_1 d_1$ and $x_2 \succ_2 d_2$.*

For an example of this framework, let us return to the “agree on an academic” example of the introduction. The problem could be modelled so that each professors outcome set X_i is the [0, 1] interval, perhaps with 0 being the least important and 1 the most important in their opinion. The agreements can be modelled as a compact subset of $[0, 1] \times [0, 1]$. Any alternative, a decision about who to employ, can be viewed as each professor receiving some point on their subjective importance scales. Some candidate is the disagreement alternative, perhaps the candidate that the Dean considers most important. This example models the pool of talent as a continuum, which is clearly an idealisation. I believe such modelling choices are both acceptable and commonly used. It would be difficult, however, to motivate the assumption that the pool of talent is a mixture set. A simpler example of this framework is the problem of dividing £100 between two people, with a disagreement alternative of each receiving zero. In this case, we can take $X_i = [£0, £100]$ for $i = 1, 2$ and let $A = \{(£x_1, £x_2) \in X_1 \times X_2 : £x_1 + £x_2 \leqslant £100\}$.

For a fixed event $E \in \mathscr{E}$, an act $x_i E y_i$ with $x_i, y_i \in X_i$ and $x_i \succcurlyeq_i y_i$ is an *E-Bet*. The set of E-Bets is $X^2_{\succcurlyeq_i}(E) := \{x_i E y_i : x_i, y_i \in X_i, x_i \succcurlyeq_i y_i\}$. It is a rank-ordered subset of the product set X_i^2, with the rank-ordering agreeing with $\succcurlyeq_i$. Each set of E-bets is endowed with the (restriction of the) product topology $\mathscr{T}_i \times \mathscr{T}_i$. Connectedness of $X^2_{\succcurlyeq_i}(E)$ follows from Wakker (1989) Lemma 7.2.

Each individual has preferences satisfying the following axioms:

A1 (Weak Ordering) Preferences for Binary Acts are complete and transitive.

A2 (Dominance) If $x_i \succ_i x'_i$ and $y_i \succ_i y'_i$ then $x_i E y_i \succ_i x'_i E y'_i$.

A3 (Continuity) For each $E \in \mathscr{E}$ and any $x_i E y_i \in X^2_{\succcurlyeq_i}(E)$, the lower and upper preference sets of E-Bets:
$\left\{x'_i E y'_i \in X^2_{\succcurlyeq_i}(E) : x_i E y_i \succ_i x'_i E y'_i\right\}$ & $\left\{x'_i E y'_i \in X^2_{\succcurlyeq_i}(E) : x'_i E y'_i \succ_i x_i E y_i\right\}$
are open.

A4 (Tradeoff Consistency) The following implication holds, where $x_i E y_i$, $x'_i E y'_i$, $z_i E y_i$ and $z'_i E y'_i$ are E-Bets, and $w_i E' x_i$, $w'_i E' x'_i$, $w_i E' z_i$ and $w'_i E' z'_i$ are E'-Bets:

$$x_i E y_i \sim_i x'_i E y'_i \,\&\, z_i E y_i \sim_i z'_i E y'_i \,\&\, w_i E' x_i \sim_i w'_i E' x'_i \Rightarrow w_i E' z_i \sim_i w'_i E' z'_i$$

The first three axioms are well known. The E-Tradeoff Consistency axiom captures the idea that we may consistently measure cardinal utility differences. It imitates the idea of strength of preferences (see Köbberling 2004), but without the need to take strengths of preferences as a primitive. The first two indifferences can be interpreted as follows: Keeping everything else fixed, replacing the first coordinate x_i with x'_i has the same effect as replacing z_i with z'_i. This reveals that the strength of preference between x_i and x'_i is the same as that between z_i and z'_i. The consistency requirement ensures that this interpretation is free of contradiction. That is, if we then find suitable w_i and w'_i to generate the third indifference, then replacing the x_i and x'_i with z_i and z'_i again should maintain the indifference.

The following theorem is due to Köbberling and Wakker (2003, p. 403). Note that separability is not required here as all events are assumed to be essential (see Wakker 1989, “Appendix A3”):

Theorem 2 (Köbberling and Wakker 2003) *Let preferences $\succcurlyeq_i$ be defined over Binary Acts and satisfy axioms A1–A4. Then there is a real-valued, increasing function u_i on X_i that is continuous in the topology $\mathscr{T}_i$ and a real-valued function $\rho_i : \mathscr{E} \to (0, 1)$ such that for any $x_i E y_i$, $x'_i E' y'_i$:*

$$
\begin{aligned}
& x_i E y_i \succcurlyeq_i x'_i E y'_i \\
& \quad \Leftrightarrow \quad \rho_i(E) u_i(x_i) + (1 - \rho_i(E)) u_i(y_i) \geqslant \rho_i(E') u_i(x'_i) + (1 - \rho_i(E')) u_i(y'_i)
\end{aligned}
$$

The function ρ is unique. The function u_i is cardinally unique.

The function u_i is called individual i's *utility function* for outcomes. The function ρ is often called a *capacity*. Referring back to the Tradeoff Consistency axiom, the implications are twofold. Firstly, we may consistently measure cardinal utility using E-Bets and secondly, it does not matter which (essential) event E we choose to take the measurements. For the Conditional Bargaining Problem, an event E is chosen in advance and will remain fixed throughout the paper.

Following Ghirardato and Marinacci (2001), preferences admitting the above representation will be called *Biseparable*. Essentially the same theory has been presented by Pfanzagl (1959, 1968), as *Generic Utility* by Miyamoto (1988) and as *Binary Rank-Dependent Utility* in Luce (1991, 2000). The renewed interest, and current popularity, of Biseparable preferences is because most of the popular models of choice coincide for rank-ordered Binary Acts, or E-Bets. Examples include the following: Expected Utility, Choquet Expected Utility (Gilboa 1987; Schmeidler 1989), Maxmin Expected Utility (Gilboa and Schmeidler 1989). A thorough survey, with many more examples, is given by Wakker (Wakker 2010: 230–231 and 298–299). Recent studies and applications of the Biseparable model include Cerreia-Vioglio et al. (2011), and Eichberger et al. (2008) studying ambiguity, and Ryan (2002) in the context of epistemic game theory.

Let $V_i(x_i E y_i) \equiv \rho_i(E) u_i(x_i) + (1 - \rho_i(E)) u_i(y_i)$. As $u_i(x_i) \equiv V_i(x_i E x_i)$, the dominance axiom ensures that preferences for outcomes are represented by u_i. Define the *certainty equivalent* function $c_i : X^2_{\succcurlyeq_i}(E) \to X_i$ as $c_i(x_i E y_i) := \{z_i \in X_i : u_i(z_i) = V_i(x_i E y_i)\}$. Given the structure and preferences here, the certainty equivalent function is a well-defined, continuous and $\succcurlyeq_i$-increasing function.

With the utilities obtained as above, we write $u(A)$ as the utility image of a feasible set A. That is, $u(A) := \{(u_1(x_1), u_2(x_2)) \in \mathbb{R}^2 : (x_1, x_2) \in A\}$. Let $M_i(A)$ be the $\succcurlyeq_i$-maximal outcome in A. Such outcomes exist as preferences are continuous and $A \subseteq X_1 \times X_2$ is compact. When the context is clear, we will write M_i and M'_i, rather than $M_i(A)$ and $M_i(A')$.

For ease of exposition, we will occasionally restrict attention to sets of Conditional Bargaining Problems that are *normal*, defined below:

Definition 3 (*Normality*) A Conditional Bargaining Problem, $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E \rangle$, is *normal* if it satisfies Structural Assumption 1 and for any $(x_1, x_2) \in A$, for both i we have $x_i \succcurlyeq_i d_i$.

To clarify, although we will assume normality at some points to ease exposition, the main theorems presented will not require this restriction.

The basic task of Bargaining Theory is to identify a feasible alternative as a solution to the problem. For a given event, E, write $\mathscr{B}_E$ as the set of all Conditional Bargaining Problems satisfying Structural Assumption 1. A *Solution* is a function, $S_E : \mathscr{B}_E \to X_1 \times X_2$, that assigns a unique, feasible alternative to any Conditional Bargaining Problem.

4 Subjective mixtures

This section outlines the notion of *Subjective Mixtures* due to Ghirardato et al. (2003) (GMMS from here on). GMMS introduced this theory in order to bring a mixture space-type structure to the purely subjective framework of Savage. In doing so, a tool was developed by which results derived in the classic Anscombe–Aumann framework can be immediately translated to that of Savage.

GMMS begin with the notion of a *Preference Average*, defined as follows:

Definition 4 (*Preference Average*) Given two outcomes $x_i \succ_i y_i$, the *Preference Average* of x_i and y_i (given E) is an outcome z_i satisfying $x_i \succ_i z_i \succ_i y_i$ and,

$$x_i E y_i \sim_i c_i(x_i E z_i) E c_i(z_i E y_i)$$

GMMS outline several justifications for the use of the term 'Preference Average'. Firstly, for any $z'_i, z''_i \in X_i$ with $x_i \succcurlyeq_i \{z'_i, z''_i\} \succcurlyeq_i y_i$ it can be shown that Biseparable preferences necessarily imply $c_i(x_i E z'_i) E c_i(z''_i E y_i) \sim_i c_i(x_i E z''_i) E c_i(z'_i E y_i)$. GMMS interpret this to mean that the inner outcomes z'_i, z''_i of the compound acts play a *symmetric* role when the individual evaluates these bets. Since I identify $x_i E x_i$ with the outcome $x_i = c_i(x_i E x_i)$ the condition may be rewritten as $c_i(x_i E x_i) E c_i(y_i E y_i) \sim_i c_i(x_i E z_i) E c_i(z_i E y_i)$. The term Preference Average is justified then observing the inner x_i and y_i play a symmetric role in the evaluation of the E-bets *and* replacing x_i and y_i with z_i retains the indifference. In short, z_i implies the kind of conditions we would expect of any general 'average' of x_i and y_i.

The second justification for the term Preference Average is seen by substituting the Biseparable representation obtained in Theorem 2. GMMS show in their Proposition 1 that z_i is a Preference Average of x_i and y_i iff:

$$u_i(z_i) = \frac{1}{2}u_i(x_i) + \frac{1}{2}u_i(y_i)$$

For the considered preferences, Preference Averages precisely identify *utility midpoints*. Note that the class of Biseparable preferences includes the following: Subjective Expected Utility preferences (Savage 1954), Choquet Expected Utility preferences (Gilboa 1987; Schmeidler 1989) and Multiple-Prior preferences (Gilboa and Schmeidler 1989). So the above holds for most of the popular models of choice under uncertainty currently used in economics. Note that Preference Averages always exist:

Lemma 5 *For any preference relation $\succcurlyeq_i$ satisfying axioms A1, A2, A3 and A4 and outcomes $x_i \succ_i y_i$, a unique Preference Average of x_i and y_i exists.*

All proofs are collected in the Appendix. Denote the Preference Average of x_i and y_i as $\frac{1}{2}x_i \oplus_i \frac{1}{2}y_i$. Then, call $\frac{1}{2}x_i \oplus_i \frac{1}{2}y_i$ a $1/2 : 1/2$ *Subjective Mixture* of x_i and y_i. It is then possible to define $\frac{3}{4}x_i \oplus_i \frac{1}{4}y_i$ as the Preference Average of x_i and $\frac{1}{2}x_i \oplus_i \frac{1}{2}y_i$. Proceeding in this way it is possible to define Subjective Mixtures for any dyadic rational and, appealing to the continuity of preferences, to construct any $\alpha : (1-\alpha)$ Subjective Mixture of x_i and y_i, denoted $\alpha x_i \oplus_i (1-\alpha)y_i$. GMMS proved the following:

Theorem 6 (Ghirardato et al. 2003) *For any preference relation $\succcurlyeq_i$ satisfying axioms A1, A2, A3 and A4:*

$$z_i = \alpha x_i \oplus_i (1-\alpha)y_i \quad \Leftrightarrow \quad u_i(z_i) = \alpha u_i(x_i) + (1-\alpha)u_i(y_i)$$

It is clear that $\alpha M_i \oplus_i (1-\alpha)d_i$ equals M_i when $\alpha = 1$ and equals d_i when $\alpha = 0$. The following monotonicity condition also follows immediately: $\alpha M_i \oplus_i (1-\alpha)d_i \succ_i \beta M_i \oplus_i (1-\beta)d_i$ whenever $\alpha > \beta$. In view of this monotonicity, the continuity of the $\oplus_i$ operation, and connectedness of X_i, every outcome $x_i \in X_i$ with $M_i \succcurlyeq_i x_i \succcurlyeq_i d_i$ is a $\alpha : 1-\alpha$ Subjective Mixture of M_i and d_i for a unique $\alpha \in [0, 1]$. For fixed M_i and d_i, given antisymmetry of preferences, the correspondence between outcomes and Subjective Mixtures is one-to-one.

The Subjective Mixture operation, $\oplus_i$, is defined over each individual's outcome set. That is, we can refer to an outcome z_i being a Subjective Mixture of two other outcomes. Define a Subjective Mixture operation, $\oplus$, over *alternatives* coordinatewise:

$$\alpha(x_1, x_2) \oplus (1-\alpha)(y_1, y_2) := (\alpha x_1 \oplus_1 (1-\alpha)y_1, \alpha x_2 \oplus_2 (1-\alpha)y_2)$$

A Conditional Bargaining Problem is *Subjective Mixture Closed* if the following implication holds:

$$\{(x_1, x_2), (y_1, y_2) \in A\} \quad \Rightarrow \quad \frac{1}{2}(x_1, x_2) \oplus \frac{1}{2}(y_1, y_2) \in A$$

Subjective mixture closedness is clearly equivalent to midpoint convexity of $u(A)$, by Theorem 6, and since $u(A)$ is compact we have:

Observation 7 *A Conditional Bargaining Problem is Subjective Mixture Closed if and only if $u(A)$ is convex.*

The condition of subjective mixture closedness is a joint restriction on the preferences of each individual and the structure of the set of alternatives. It is, essentially, a testable hypothesis. It does, however, hold naturally in certain important cases. For example, the problem of dividing £100 between two people with a disagreement alternative of each receiving zero. In this case we took $X_i = [£0, £100]$ for $i = 1, 2$ and let $A = \{(£x_1, £x_2) \in X_1 \times X_2 : £x_1 + £x_2 \leqslant £100\}$. A Conditional Bargaining Problem with this set of alternatives and disagreement will be Subjective Mixture Closed if the individuals have utilities u_1 and u_2 that are concave. "Utility that is concave in money" seems to be more empirically relevant than "utility that is linear in probabilities".

A weaker condition than subjective mixture closedness is *Comprehensiveness*. A Conditional Bargaining Problem $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$ is d-Comprehensive if $y_i \succcurlyeq_i$

$x_i \succcurlyeq_i d_i$ for $i = 1, 2$ and $y \in A$ jointly imply $x \in A$. The *Comprehensive hull* of a set A with respect to an alternative d, denoted $comp[A; d]$, is the smallest d-Comprehensive set containing A. The following observation follows from elementary substitution of the preference functionals:

Observation 8 *A Conditional Bargaining Problem is d-Comprehensive if and only if $u(A)$ is $u(d)$-Comprehensive.*

We will make use of the following notation:

$$\Psi_1(\alpha; A, \oplus_1, \oplus_2) := \{\beta \in [0, 1] : (\beta M_1 \oplus_1 (1 - \beta)d_1, \alpha M_2 \oplus_2 (1 - \alpha)d_2) \in A\}$$

So $\Psi_1(\alpha; A, \oplus_1, \oplus_2)$ is the set of β's such that a $\beta : 1 - \beta$ Subjective Mixture of individual $1'$s best and disagreement outcomes is feasible when 2 has an $\alpha : 1 - \alpha$ Subjective Mixture of their best and disagreement outcomes. $\Psi_2(\alpha; A, \oplus_1, \oplus_2)$ is defined similarly.

5 Applications

5.1 The Conditional Nash Solution

As a first application of Subjective Mixtures to Conditional Bargaining Problems, this section provides a preference foundation for a Conditional version of the Bargaining Solution of Nash (1950). For a Bargaining Problem, $\langle U, a\rangle$, the Nash Bargaining Solution, $\mathscr{N}$, is the feasible point $(\lambda_1, \lambda_2) \in U$ such that:

$$(\lambda_1, \lambda_2) = \arg\max_{(\beta_1,\beta_2)\geqslant(a_1,a_2)}\{(\beta_1 - a_1)(\beta_2 - a_2)\}$$

The Nash Bargaining Solution maximises the product of positive utility differences from the disagreement point. Although the formula itself is simple, the behavioural content of the solution is not immediately clear. An interpretation of the Nash Bargaining Solution, in the context of risky lotteries, has been provided by Rubinstein et al. (1992), and we will discuss a version of this in what follows.

Nash considered the following axioms for a Bargaining Solution S, defined over the class of compact and convex Bargaining Problems:

Axiom S1 (Pareto Efficiency): If $S(\langle U, a\rangle) = (\alpha, \beta)$ then there is no $(\alpha', \beta') \in U$ such that $\alpha' \geqslant \alpha$, $\beta' \geqslant \beta$ and $(\alpha', \beta') \neq (\alpha, \beta)$.

Axiom S2 (Symmetry): If $\langle U, a\rangle$ is such that $a_1 = a_2$, and for all $\alpha \in \mathbb{R}$ we have $\Phi_1(\alpha; U) = \Phi_2(\alpha; U)$, then $S_1(\langle U, a\rangle) = S_2(\langle U, a\rangle)$.

Axiom S3 (Contraction Independence): If $V \subseteq U$ and $S(\langle U, a\rangle) \in V$ then $S(\langle U, a\rangle) = S(\langle V, a\rangle)$

Axiom S4 (Cardinal Invariance): If $\langle U', a'\rangle$ is obtained from $\langle U, a\rangle$ by an affine transformation $u'_i = \gamma_i u_i + \delta_i$, $\gamma_i > 0$, $\delta_i \in \mathbb{R}$, $i = 1, 2$, then $S_i(\langle U', a'\rangle) = \gamma_i S_i(\langle U, a\rangle) + \delta_i$.

For a discussion of these axioms see Thomson (2009). The interpretation of Contraction Independence is, keeping the disagreement point fixed, when a solution is chosen out of a large set, and a smaller subset contains this solution, then the solution still holds for the smaller problem. Presumably, those points not chosen in the larger problem are already excluded from consideration when the smaller problem is examined. We will discuss this more in what follows. Nash (1950) proved the following:

Theorem 9 (Nash 1950) *A Bargaining Solution, defined over the class of compact and convex Bargaining Problems, is the Nash Solution if and only if it satisfies the Pareto Efficiency, Symmetry, Cardinal Invariance and Contraction Independence axioms (S1, S2, S3 and S4).*

In what follows, we will obtain a version of the Nash Solution for Conditional Bargaining Problems. Preference axioms that characterise the solution will be presented. It will be useful to first obtain a new axiomatisation of the Nash Solution. This is because the Cardinal Invariance axiom becomes, essentially, redundant as a preference condition. By itself, all Cardinal Invariance does is allow us to meaningfully discuss utility axioms. However, if we attempt to simply translate the remaining axioms into preference conditions, we will not get far. This is because Cardinal Invariance, although redundant by itself, actually adds empirical content to the other axioms. So, the important step is to remove Cardinal Invariance from the axiom set. Then, restate the remaining axioms, but embedding the preference implications of Cardinal Invariance in each. I therefore suggest the following:

Axiom S2* (Cardinal Utility Symmetry): If, for a problem $\langle U, a\rangle$, there exists some affine transformations of the utilities, $u'_i = \gamma_i u_i + \delta_i$, with $\gamma_i > 0, \delta_i \in \mathbb{R}, i = 1, 2$, such that:
 i. $a'_1 = a'_2$
 ii. For all $\alpha \in \mathbb{R}$, $\Phi_1(\alpha; U') = \Phi_2(\alpha; U')$
then $\gamma_1 S_1(\langle U, a\rangle) + \delta_1 = \gamma_2 S_2(\langle U, a\rangle) + \delta_2$.

The Cardinal Utility Symmetry axiom states that when a Bargaining Problem can be transformed, by positive affine transformations of the utilities, such that the problem becomes symmetric, then the solution to the original problem will map to a symmetric point. The Cardinal Utility Symmetry axiom implies the Symmetry axiom. To see this, simply restrict the permissible transformations to be the identity map. Similarly, amend Contraction Independence as follows:

Axiom S3* (Cardinal Contraction Independence): For the problems $\langle U, a\rangle$ and $\langle V, b\rangle$, if there exists some affine transformations of the utilities $u'_i = \gamma_i u_i + \delta_i$ and $v'_i = \gamma'_i v_i + \delta'_i$, with $\gamma_i, \gamma'_i > 0, \delta_i, \delta'_i \in \mathbb{R}$ and $i = 1, 2$, such that:
 i. $a' = b'$
 ii. $V' \subseteq U'$
 iii. $(\gamma_1 S_1(\langle U, a\rangle) + \delta_1, \gamma_2 S_2(\langle U, a\rangle) + \delta_2) \in V'$
Then,

$$(\gamma_1 S_1(\langle U, a\rangle)+\delta_1, \gamma_2 S_2(\langle U, a\rangle)+\delta_2)=(\gamma'_1 S_1(\langle V, b\rangle)+\delta'_1, \gamma'_2 S_2(\langle V, b\rangle)+\delta'_2)$$

Cardinal Contraction Independence requires that if two Bargaining Problems can be affinely transformed so that: the transformed disagreement points coincide, one transformed problem contains the other, and the solution to the first is mapped to a point contained in the other transformed problem, then the solution to the first problem is mapped to the same point as the solution to the other problem. Of course, by restricting the permissible transformations to the identity map, we see that Cardinal Contraction Independence implies Contraction Independence. We can now state our axiomatisation of the Nash Bargaining Solution:

Theorem 10 *A Bargaining Solution, defined over the class of compact and convex problems, is the Nash Solution if and only if it satisfies the Pareto Efficiency, Cardinal Utility Symmetry and Cardinal Contraction Independence axioms (S1, S2* and S5*).*

The proof of Theorem 10 is presented in "Appendix A.2". Theorem 10 is useful because translating Cardinal Contraction Independence S5* to a preference axiom in the Conditional Bargaining Problem framework is simplified.

To define a Nash-type Solution for Conditional Bargaining Problems, I adapt the *Ordinal Nash Solution* of Rubinstein et al. (1992), RST. The *Conditional Nash Solution* is defined as follows:

Definition 11 A Conditional Nash Solution, for a Conditional Bargaining Problem $\langle A, d, \succsim_1, \succsim_2, E\rangle$, is an alternative (y_1, y_2) satisfying the following condition:

$$\left\{\begin{array}{c} \alpha \in [0,1],\ (x_1, x_2) \in A, \\ \alpha x_i \oplus_i (1-\alpha) d_i \succ_i y_i \end{array}\right\} \Rightarrow x_j \precsim_j \alpha y_j \oplus_j (1-\alpha) d_j$$

The interpretation of the Conditional Nash Solution is as follows. Suppose the candidate (y_1, y_2) is on the table and, for the sake of clarity, that $\alpha = 1/2$. Individual i views his proposed outcome as 'not even half as good' as another and suggests this instead. The other individual sees the newly suggested outcome as 'not even half as good' as the one on the table. As neither individual's claim is stronger, in terms of intensity of preference, changing from the alternative on the table could be deemed 'unfair' to at least one individual. Once a Conditional Nash Solution is on the table, all other alternatives can be similarly dismissed. Whenever this solution is well defined and preferences are Biseparable, it follows by a reasoning similar to Proposition 1 of RST that the solution is the alternative which maximises the product of utility differences from the disagreement alternative. The following observation follows from elementary substitution of the preference functionals:

Observation 12 *If preferences are Biseparable, then an alternative $(y_1, y_2) \in A$ is a Conditional Nash Solution for a Conditional Bargaining Problem $\langle A, d, \succsim_1, \succsim_2, E\rangle$ if and only if:*

$$(u_1(y_1), u_2(y_2)) = \arg\max_{x_1 \succsim_1 d_1, x_2 \succsim_2 d_2} \{(u_1(x_1) - u_1(d_1))(u_2(x_2) - u_2(d_2))\}$$

We will now obtain a characterisation of the Conditional Nash Solution. That is, we will consider an arbitrary Conditional Bargaining Solution, S, and restrict S by imposing three axioms: *Pareto Efficiency*, *Subjective Symmetry* and *Subjective Contraction Independence*. It will be shown that an arbitrary solution S satisfying these axioms is equivalent to S being the Conditional Nash Solution. The Pareto Efficiency axiom is already known, being a simple translation of axiom S1:

Axiom P1 (Pareto Efficiency) If $S(\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle) = (x_1, x_2)$, then there is no $(y_1, y_2) \in A$ such that, for $i \neq j$, $y_i \succcurlyeq_i x_i$ and $y_j \succ_j x_j$.

My second axiom, Subjective Symmetry, plays a similar role to Cardinal Utility Symmetry. The Cardinal Utility Symmetry axiom involves comparisons of utility numbers between the two individuals. This is perfectly acceptable, it is simply comparing numbers. It is not immediately clear, however, what is implied for the underlying preferences because the utilities are not uniquely determined. To state an axiom for preferences that is equivalent to Cardinal Utility Symmetry, it is useful to have an appropriate unit of currency for interpersonal comparisons. This is where Subjective Mixtures are particularly useful, and I use them to define the axiom.

Axiom P2 (Subjective Symmetry): If, for a Conditional Bargaining Problem $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$, the set of alternatives A is Subjectively Symmetric, that is, it is such that, for any $\alpha \in [0, 1]$, $\Psi_1(\alpha; A, \oplus_1, \oplus_2) = \Psi_2(\alpha; A, \oplus_1, \oplus_2)$, then the solution $S(\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle)$ is also Subjectively Symmetric.

The interpretation of Subjective Symmetry is as follows. A Conditional Bargaining Problem is Subjectively Symmetric if: Whenever there is a feasible alternative where individual 1 receives an α, and 2 receives a β Subjective Mixture of the best and disagreement outcomes respectively, then there is a feasible alternative that gives 1 a β and 2 an α Subjective Mixture of the best and disagreement outcomes, respectively. The Subjective Symmetry axiom demands that, when the Conditional Bargaining Problem is Subjectively Symmetric, the solution should also be Subjectively Symmetric. That is, each should receive an outcome that uses the same λ Subjective Mixture. The following lemma provides a preference foundation for the Cardinal Utility Symmetry condition:

Lemma 13 *If preferences are Biseparable then a Conditional Bargaining Solution, defined over the class of normal Conditional Bargaining Problems, satisfies Subjective Symmetry (P2) if and only if it satisfies Cardinal Utility Symmetry (S2*) with respect to the utilities concerned.*

The final axiom introduced in this section is the following:

Axiom P3 (Subjective Contraction Independence): If, for two Conditional Bargaining Problems $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$ and $\langle A', d', \succcurlyeq_1, \succcurlyeq_2, E\rangle$, the following conditions hold:

i. For all $\alpha \in [0, 1]$, $i = 1, 2$, $\Psi_i(\alpha; A', \oplus_1, \oplus_2) \subseteq \Psi_i(\alpha; A, \oplus_1, \oplus_2)$.
ii. $S(\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle) = (\lambda_1 M_1 \oplus_1 (1 - \lambda_1)d_1, \lambda_2 M_2 \oplus_2 (1 - \lambda_2)d_2)$
iii. $(\lambda_1 M'_1 \oplus_1 (1 - \lambda_1)d'_1, \lambda_2 M'_2 \oplus_2 (1 - \lambda_2)d'_2) \in A'$

Then $S(\langle A', d', \succcurlyeq_1, \succcurlyeq_2, E\rangle) = (\lambda_1 M'_1 \oplus_1 (1 - \lambda_1)d'_1, \lambda_2 M'_2 \oplus_2 (1 - \lambda_2)d'_2)$.

Call one Conditional Bargaining Problem a *Subjective Contraction* of another if: Whenever the problem contains an alternative where each receives some Subjective

Mixture of their best and disagreement outcomes in that problem, then the other problem contains an alternative where the same Subjective Mixtures of (possibly different) best and disagreement outcomes are used. The Subjective Contraction Independence axiom insists that if one problem is a Subjective Contraction of another, but still contains an alternative where each receives the same Subjective Mixture of best and disagreement outcomes as the other problem's solution, then that alternative is the solution to the problem. The axiom does not necessarily imply the two solutions are the same alternative, only that they each use the same Subjective Mixture values of each problem's best and disagreement outcomes.

An immediate consequence of the Subjective Contraction Independence axiom is that the solution is now independent of the alternatives dominated by the disagreement alternative. For ease of exposition, we will therefore suppose that the disagreement outcomes are always each individual's worst outcome. This is without loss of generality, as anything below the disagreement alternative can be deleted without affecting the solution.

The simplest case where conditions *i.* and *ii.* hold is when $A \subseteq A'$ and, further, that $M = M'$ and $d = d'$. In this case, we could call the contraction *objective* rather than subjective. The Subjective Contraction Independence axiom, however, still has power when A and A' are disjoint. Such sets can be compared by examining which Subjective Mixtures are feasible in each. When conditions i. and ii. hold, and preferences are Biseparable, there will always exist utilities such that the images of the feasible sets are related by set inclusion. But, because the underlying sets of alternatives may be disjoint, the name Independence of Irrelevant Alternatives is particularly inappropriate here. As the following lemma proves the equivalence of Subjective Contraction Independence and Cardinal Contraction Independence, the latter implying Contraction Independence, the comments above apply to that condition as well. Hence I also offer an improved understanding of a known condition:

Lemma 14 *If preferences are Biseparable, then a Conditional Bargaining Solution, defined over the class of normal Conditional Bargaining Problems, satisfies Subjective Contraction Independence (P3) if and only if it satisfies Cardinal Contraction Independence (S3*) with respect to the utilities concerned.*

This section concludes with a preference foundation for the Conditional Nash Solution.

Theorem 15 *A Conditional Bargaining Solution, defined over the class of Subjective Mixture Closed Problems satisfying Structural Assumption 1, with each individual having Biseparable preferences, is the Conditional Nash Solution if and only if it satisfies the Pareto Efficiency, Subjective Symmetry and Subjective Contraction Independence axioms (P1, P2 and P3).*

The use of GMMS's Subjective Mixtures has allowed us to obtain a characterisation of the Conditional Nash Solution. The use of Subjective Mixtures was necessary both to interpret our restated axioms into preference conditions and to define the required Subjective Mixture Closed structures.

5.2 The Conditional Kalai–Smorodinsky Solution

This section provides a new foundation for the solution of Kalai and Smorodinsky (1975), as applied to Conditional Bargaining Problems. We begin with the Kalai–Smorodinsky result in the context of Bargaining Problems. As an interim result, I will amend the Kalai–Smorodinsky axioms and provide a new foundation for the Kalai–Smorodinsky Solution. This result paves the way for a preference foundation, derived using Subjective Mixtures.

For a Bargaining Problem, $\langle U, a\rangle$, We will use the notation $B_i(\langle U, a\rangle)$ to denote the maximal payoff for i in the set $\{\alpha \in U : \alpha \geqslant a\}$. This is well defined as U is compact. When the context is clear, we will simply write B_i. The *utopia point* is $B = (B_1, B_2)$.

Kalai and Smorodinsky (1975) provided a characterisation of their solution over the set of compact and convex problems. Following Conley and Wilkie (1991), we will assume Comprehensiveness rather than convexity. Four axioms will characterise the solution. Firstly, on the domain of comprehensive problems, we need to replace Pareto Efficiency with *Weak Pareto Efficiency*:

Axiom S1* (Weak Pareto Efficiency): If $S(\langle U, a\rangle) = (\alpha, \beta)$ then there is no $(\alpha', \beta') \in U$ such that $\alpha' > \alpha$ and $\beta' > \beta$.

See Theorem 1 of Conley and Wilkie (1991) for a simple explanation for why S1 must be replaced with S1*. The Symmetry and Cardinal Invariance axioms, outlined in the previous section, are also assumed. The final axiom, *Individual Monotonicity*, is the following:

Axiom S5 (Individual Monotonicity): If $\langle U, a\rangle$ and $\langle V, b\rangle$ are such that $U \subseteq V$, $B_i(\langle U, a\rangle) = B_i(\langle V, b\rangle)$, $i \neq j$ and $a = b$, then $S_j(\langle U, a\rangle) \leq S_j(\langle V, b\rangle)$.

The Kalai–Smorodinsky Solution, $\mathscr{K}$, is the efficient point (α_1, α_2) such that, $(\alpha_1, \alpha_2) = \lambda B + (1 - \lambda)a$ for some $\lambda \in [0, 1]$. The representation is simple: connect the disagreement point a and utopia point B with a line and then select the efficient point on this line. For the class of Bargaining Problems considered, $\mathscr{K}$ is well defined and unique. Conley and Wilkie (1991) proved the following:

Theorem 16 (Conley and Wilkie 1991) *A Bargaining Solution, defined over the class of compact and a-comprehensive Bargaining Problems, is the Kalai–Smorodinsky Solution if and only if it satisfies the Weak Pareto Efficiency, Symmetry, Cardinal Invariance axioms and Individual Monotonicity axioms (S1*, S2, S4 and S5).*

As with the axiomatisation of the Nash Solution, before considering a preference foundation, I first restate Kalai and Smorodinsky's axioms and provide a new characterisation of their Solution. This interim result proceeds along similar lines to the previous section. Remove the Cardinal Invariance axiom, S4, which is redundant when discussing preferences. We have already obtained Cardinal Utility Symmetry S2* in the previous section. Now adapt the Individual Monotonicity axiom, in a way that retains the empirically meaningful consequences lent by Cardinal Invariance, as follows:

Axiom S5* (Cardinal Utility Monotonicity): If, for the problems $\langle U, a\rangle$ and $\langle V, b\rangle$, there exists some affine transformations of the utilities $u'_k = \gamma_k u_k + \delta_k$ and $v'_k = \gamma'_k v_k + \delta'_k$, with $\gamma_k, \gamma'_k > 0$, $\delta_k, \delta_k \in \mathbb{R}$, and $k = 1, 2$, such that:

i. $a' = b'$.
ii. For all $\alpha \in \mathbb{R}$, $\Phi_i(\alpha; V') = \Phi_i(\alpha; U')$.
iii. For all $\alpha \in \mathbb{R}$, $\Phi_j(\alpha; V') \supseteq \Phi_j(\alpha; U')$.

Then $\gamma'_j S_j(\langle V, b\rangle) + \delta'_j \geqslant \gamma_j S_j(\langle U, a\rangle) + \delta_j$.

The Cardinal Utility Monotonicity axiom states: If two Bargaining Problems can be transformed, by (possibly distinct) positive affine transformations of the utilities, so that the conditions of the Individual Monotonicity axiom apply to the transformed problems, then the solution utility levels for individual j in each of the original problems will map to points that preserve the inequality demanded by Individual Monotonicity. Similar to above, the Cardinal Utility Monotonicity axiom implies Individual Monotonicity. We now have the following axiomatisation of the Kalai–Smorodinsky Solution:

Theorem 17 *A Bargaining Solution, defined over the class of compact and a-comprehensive Bargaining Problems, is the Kalai–Smorodinsky Solution if and only if it satisfies the Weak Pareto Efficiency, Cardinal Utility Symmetry and Cardinal Utility Monotonicity axioms (S1*, S2* and S5*).*

The proof of Theorem 17 is presented in "Appendix A.6". Theorem 17 is useful because the step from axioms S1*, S2* and S3* to preference axioms in the Conditional Bargaining Problem framework is simplified.

Recall the intuitive content of the Kalai–Smorodinsky Solution, $\mathscr{K}$. Each individual is receiving the same proportions of their best and their disagreement utilities. The use of Subjective Mixtures allows us to express this directly as a preference condition: each individual receives the same Subjective Mixture of their most preferred and their disagreement outcomes. Define the *Conditional Kalai–Smorodinsky Solution* as follows:

Definition 18 (*Conditional Kalai–Smorodinsky Solution*) A Conditional Kalai–Smorodinsky Solution, for a Conditional Bargaining Problem $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$, is an efficient alternative (y_1, y_2) such that, for some $\lambda \in [0, 1]$:

$$(y_1, y_2) = \lambda(M_1, M_2) \oplus (1 - \lambda)(d_1, d_2)$$

A straightforward substitution of the preference functionals, for each individual, leads to the following elementary observation:

Observation 19 *If preferences are Biseparable, then* (y_1, y_2) *is a Conditional Kalai-Smorodinsky Solution if and only if it is an efficient alternative and:*

$$(u_1(y_1), u_2(y_2)) = \lambda(u_1(M_1), u_2(M_2)) + (1 - \lambda)(u_1(d_1), u_2(d_2))$$

I will provide a Subjective Mixture foundation for the Conditional Kalai–Smorodinsky Solution. We have already obtained P2* as preference foundations for axioms S2*, respectively. Translating S1* to a preference condition P1* is straightforward. Finally, I now suggest the following axiom:

Axiom P5 (Subjective Monotonicity): If, for two Conditional Bargaining Problems $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$ and $\langle A', d', \succcurlyeq_1, \succcurlyeq_2, E\rangle$, the following conditions hold:

i. For all $\alpha \in [0, 1]$, $\Psi_i(\alpha; A, \oplus_1, \oplus_2) = \Psi_i(\alpha; A', \oplus_1, \oplus_2)$.
ii. For all $\alpha \in [0, 1]$, $\Psi_j(\alpha; A, \oplus_1, \oplus_2) \supseteq \Psi_j(\alpha; A', \oplus_1, \oplus_2)$
iii. $S_j(\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle) = \lambda M_j \oplus_j (1 - \lambda)d_j$.
iv. $S_j(\langle A', d', \succcurlyeq_1, \succcurlyeq_2, E\rangle) = \lambda' M'_j \oplus_j (1 - \lambda')d'_j$.

Then $\lambda \geqslant \lambda'$.

In words, the Subjective Monotonicity axiom is as follows. If two Conditional Bargaining Problems are such that: Firstly, the same set of Subjective Mixtures of one individual's (possibly different) best and disagreement outcomes are available in each feasible set. Secondly, the feasible set of one problem has at least as many Subjective Mixtures of best and disagreement outcomes for the other individual, and this individual receives a λ and a λ' Subjective Mixture in the larger and smaller problem, respectively. Then, the solution to the larger problem offers this individual a better Subjective Mixture.

As with Subjective Contraction Independence, an immediate consequence of the Subjective Monotonicity axiom is that the solution is now independent of the alternatives dominated by the disagreement alternative. Suppose two problems have feasible sets that are the same above the disagreement alternative, but are otherwise different. Then conditions i. and ii. automatically hold. If the solutions are different, a contradiction of Subjective Monotonicity (applied twice) will always result. For ease of exposition we suppose, without loss of generality, that the problems are normal.

The simplest case is when i. and ii. hold, and further that $M = M'$ and $d = d'$. Then the axiom requires that the solution does not get worse for one individual as the feasible set of alternatives increases in a way favourable only to him. The axiom does, however, imply more than this. The feasible sets of each problem could even be disjoint, yet the axiom still has power. As Subjective Mixtures are used, it makes perfect sense to discuss two feasible sets as having more available for one individual, even if the feasible sets cannot be compared on the basis of set inclusion. In fact, if conditions *i.* and *ii.* hold, and preferences are Biseparable, then there will always be utilities such that the images of A' and A can be compared by the typical set inclusion approach used by Kalai and Smorodinsky's Individual Monotonicity axiom. The following lemma clarifies this and provides the necessary preference foundation for the Cardinal Monotonicity axiom:

Lemma 20 *If preferences are Biseparable then a Conditional Bargaining Solution, defined over the class of normal Conditional Bargaining Problems, satisfies Subjective Monotonicity (P5) if and only if it satisfies Cardinal Utility Monotonicity (S5*) with respect to the utilities concerned.*

This section concludes with a preference foundation for the Conditional Kalai–Smorodinsky Solution.

Theorem 21 *A Conditional Bargaining Solution, defined over the class of d-Comprehensive Problems satisfying Structural Assumption 1, with each individual having Biseparable preferences, is the Conditional Kalai–Smorodinsky Solution if and only* *if it satisfies the Weak Pareto Efficiency, Subjective Symmetry and Subjective Monotonicity axioms (P1*, P2 and P5).*

6 Separating the behavioural from the technical

This section will address an issue with how I have used of Subjective Mixtures. The Subjective Mixture axioms above have included the use of Subjective Mixtures across the entire [0, 1] interval. Some Subjective Mixture values are easily observable. For other values it may take many observations, although the number of measurements is finite. In either case, the conditions are falsifiable by observing preferences, and we will call such conditions *behavioural*. Unfortunately, there are Subjective Mixture values, such as a 1/3:2/3 mixture, that can only be verified approximately. Precise elicitation of such values would require an infinite number of observations, if we were to use iterations of preference averages. Conditions of this type, which are not readily falsifiable, we will call *technical*. Continuity of preferences, for example, is an often assumed technical condition.

Although technical assumptions are often necessary, it is desirable to separate the axiom set into those that are purely behavioural and those that are technical. In this sense, the axioms that have verifiable, empirical content may be assessed.[1]

I now offer a solution to this problem. The first step is to restrict our Subjective Mixture axioms (Subjective Symmetry, Subjective Monotonicity and Subjective Contraction Independence) to hold only when the Subjective Mixtures involved are *dyadic rationals*. The set of dyadic rationals is $\mathscr{D} := \{\beta \in [0,1] : \beta = \sum_{i=1}^{N} a_i/2^i, a_i \in \{0,1\}, N \in \mathbb{N}\}$. For example, rewriting Subjective Symmetry:

Axiom P2* (Restricted Subjective Symmetry): If, for a Conditional Bargaining Problem
$\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$, the following hold:
i. For any $\alpha \in \mathscr{D}$, $\Psi_1(\alpha; A, \oplus_1, \oplus_2) = \Psi_2(\alpha; A, \oplus_1, \oplus_2)$ and,
ii. $S(\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle) = (\lambda_1 M_1 \oplus_1 (1-\lambda_1)d_1, \lambda_2 M_2 \oplus_2 (1-\lambda_2)d_2)$, where $\lambda_1, \lambda_2 \in \mathscr{D}$,
then $\lambda_1 = \lambda_2$.

It is straightforward to rewrite Subjective Contraction Independence and Subjective Monotonicity in a way that refers only to dyadic Subjective Mixtures. Since any dyadic rational may be reached using finitely many Preference Average iterations, the axioms restricted in this way are what I call behavioural.

The second step is to formulate a continuity condition. Let $\mathscr{A}$ be the set of all compact subsets of $X_1 \times X_2$. These are the feasible sets of our Conditional Bargaining Problems. We may regard $\mathscr{A}$ as a topological space in its own right. In particular, endow $\mathscr{A}$ with the *Topology of Closed Convergence* $\mathscr{T}_c$ (see, for example, Aliprantis and Border 1999, pp. 119–123). Let $\{A_k\}$ denote a sequence of feasible sets. Write $\{\langle A_k, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle\} \to \langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$ whenever $\{A_k\} \to A$ in the Topology

[1] Be warned, however, that one should judge axiom *sets* rather than axioms in isolation. we have already discussed Cardinal Invariance, in this respect. An excellent example, regarding the interaction of continuity of preferences with other axioms, is given by Wakker (1996, p. 225).

of Closed Convergence. Note that all components of the problem are kept fixed, except the set of feasible alternatives. The solution S is $\mathscr{T}_c$-continuous if $\{\langle A_k, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle\} \to \langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$ implies $\{S(\langle A_k, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle)\} \to S(\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle)$. The following axiom insists that a Conditional Bargaining Solution be continuous in this respect:

Axiom P6 (Continuity): S is $\mathscr{T}_c$-continuous over $\mathscr{A}$.

An intuitive interpretation of the Continuity axiom is that small changes of the problem do not result in large changes in the solution. The Continuity axiom is the technical condition that extends our restricted, behavioural axioms to the full force axioms previously stated. The following theorems hold:

Theorem 22 *A Conditional Bargaining Solution, defined over the class of Subjective Mixture Closed Problems satisfying Structural Assumption 1, with each individual having Biseparable preferences, is the Conditional Nash Solution if and only if it satisfies Pareto Efficiency, the restricted Subjective Symmetry and Subjective Contraction Independence axioms, and Continuity.*

Theorem 23 *A Conditional Bargaining Solution, defined over the class of d-Comprehensive Problems satisfying Structural Assumption 1, with each individual having Biseparable preferences, is the Conditional Kalai–Smorodinsky Solution if and only if it satisfies Weak Pareto Efficiency, the restricted Subjective Symmetry and Subjective Monotonicity axioms, and Continuity.*

If we restrict the Subjective Mixture axioms to dyadic rationals, we can be assured that we are dealing with behavioural conditions. Unfortunately, it may still take a great many observations to elicit certain values. Therefore, finding an equivalent axiom set that uses *only preference averages* would constitute a significant improvement. I present this problem for future research.

7 Closing comments

The aim of this paper was to find an approach to solving the Bargaining Problem that does not rely on the use of risky lotteries. To achieve this, I proposed the notion of a Conditional Bargaining Problem. This extension of a Bargaining Problem allows the cardinal information about preferences to be measured. Specifically, we may apply the toolkit developed under the title of Conjoint Measurement (Krantz et al. 1971). The theory of Conjoint Measurement has been extensively developed as part of the non-Expected Utility agenda.

This paper considered single, two-person bargaining problems. An interesting case arises when the individuals are engaging in two or more separate problems, as in Peters (1986). In that case, a conjoint structure arises naturally, without the need to extend the problem. A topic for further research is to exploit that structure to derive a simultaneous characterisation of the cardinal utilities and solutions within such a riskless setting.

The Subjective Mixture methods of Ghirardato et al. (2003) are especially powerful when applied to our approach. Firstly, Subjective Mixtures allowed us to use a mixture space-type framework, where one was not naturally provided. Secondly, the use of Subjective Mixtures provided us with a currency that both reflects individual preferences and, since they are uniquely determined, allows us to make interpersonal comparisons. It also becomes possible to compare feasible sets where no obvious set inclusion relation exists. Subjective Mixtures were used at every stage: to define the structural assumptions (Subjective Mixture Closedness), to define axioms (Subjective Symmetry, Subjective Monotonicity, Subjective Contraction Independence) and also to state the preference content of the solutions.

The Subjective Mixture techniques used in this paper begin with the notion of a Preference Average. Section 4 outlined the definition of a Preference Average. The main justification for the term Preference Average was that such outcomes are *utility midpoints*. Earlier, Vind (1987, 1991) had derived and studied such an operation through observed preferences (see also Vind 2003, Chapter 7). There are several other techniques for measuring utility midpoints. Most recently, Baillon et al. (2009) presented a particularly simple method. Starting with their method, or any of the references contained there concerning utility midpoint elicitation, one can proceed to construct Subjective Mixtures as described in Sect. 4. It should also be noted that utility midpoints have been successfully elicited in several individual choice experiments (Wakker and Deneffe 1996; Abdellaoui et al. 2007). In this sense, a conjoint extension approach to solving Bargaining Problems relies on a proven technology.

Appendix

A.1 Proof of Lemma 5

Proof Preferences over X_i^2 are represented by a continuous function V_i. Fix any $x_i \succ_i y_i$ and define a function f so that $f(t) = V_i(c_i(x_i E t) E c_i(t E y_i))$ for all $t \in X_i$. f is clearly continuous, being the composition of continuous functions. By the dominance axiom, $f(x) > V_i(x_i E y_i) > f(y)$. Then, since f is continuous on a connected set X_i, there is a z_i so that $f(z_i) = V_i(x_i E y_i)$ equivalent to the sought after indifference. One can show $x_i \succ_i z_i \succ_i y_i$ and that z_i is unique using the antisymmetry of preferences and the dominance axiom. □

A.2 Proof of Theorem 10

Let S be a Bargaining Solution satisfying axioms S1, S2* and S5*. Let $\mathcal{N}$ denote the Nash Solution. That $\mathcal{N}$ satisfies S1, S2* and S5* is elementary. We already know $\mathcal{N}$ is well defined and satisfies S1, S2, S3 and S5. We will show $S = \mathcal{N}$.

If a Bargaining Problem $\langle U, a\rangle$ is Cardinal Utility Symmetric then it can be affinely transformed to a symmetric problem $\langle \gamma U + \delta, \gamma a + \delta\rangle$. By S1 and S2*, the solution $S(\langle U, a\rangle)$ maps to an efficient symmetric point. Such a point will also be selected by $\mathcal{N}$, so:

$$\begin{aligned}\gamma_1 S_1(\langle U, a\rangle) + \delta_1 &= \gamma_2 S_2(\langle U, a\rangle) + \delta_2\\ &= \mathcal{N}_1(\langle \gamma U + \delta, \gamma a + \delta\rangle) = \mathcal{N}_2(\langle \gamma U + \delta, \gamma a + \delta\rangle)\end{aligned}$$

Since $\mathcal{N}$ satisfies S3, $\mathcal{N}_i(\langle\gamma U+\delta, \gamma a+\delta\rangle) = \gamma_i \mathcal{N}_i(\langle U, a\rangle) + \delta_i$ for $i = 1, 2$. So, $S(\langle U, a\rangle) = \mathcal{N}(\langle U, a\rangle)$.

Now suppose a Bargaining Problem $\langle U, a\rangle$ is not Cardinal Utility Symmetric. Apply a positive affine transformation such that $\mathcal{N}_i(\langle\gamma U+\delta, \gamma a+\delta\rangle) = 1$ and $\gamma_i a_i + \delta_i = 0$ for $i = 1, 2$. Since $\mathcal{N}$ maximises the product $\alpha_1\alpha_2$, and the problem is convex, we must have $\alpha_1+\alpha_2 \leqslant 2$ for any $(\alpha_1, \alpha_2) \in \gamma U+\delta$. Let $U' = \{(\alpha_1, \alpha_2) \in \mathbb{R}^2_+ : \alpha_1+\alpha_2 \leqslant 2\}$. The Bargaining Problem $\langle U', (0, 0)\rangle$ is Cardinal Utility Symmetric (take $\gamma_i = 1$ and $\delta_i = 0$ for $i = 1, 2$) so axioms S1 and S2* imply:

$$S_1(\langle U', (0, 0)\rangle) = S_2(\langle U', (0, 0)\rangle) = 1$$

Since $\gamma U + \delta \subseteq U'$ we may apply axiom S5* to get, for $i = 1, 2$:

$$\gamma_i S_i(\langle U, a\rangle) + \delta_i = S_i(\langle U', (0, 0)\rangle) = 1$$

Finally, we have:

$$\gamma_i S_i(\langle U, a\rangle) + \delta_i = \mathcal{N}_i(\langle\gamma U, \gamma a + \delta\rangle) = \gamma_i \mathcal{N}_i(\langle U, a\rangle) + \delta_i$$

So $S(\langle U, a\rangle) = \mathcal{N}(\langle U, a\rangle)$.

A.3 Proof of Lemma 13

Proof Substituting the preference functionals, it is clear that a normal Conditional Bargaining Problem $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$ is Subjectively Symmetric if and only if there are utilities v_1 and v_2 such that $v_1(M_1) = v_2(M_2) = 1$ and $v_1(d_1) = v_2(d_2) = 0$ such that, for all $\alpha \in [0, 1]$, $\Phi_1(\alpha; v(A)) = \Phi_2(\alpha; v(A))$. The Subjective Symmetry axiom then selects the alternative (y_1, y_2) such that $v_1(y_1) = v_2(y_2) = \lambda$. Clearly, any other representation using appropriate utilities can be affinely transformed to coincide with this normalised representation. The alternative selected by the axiom remains the same if and only if the transformation of any other representation maps the solution to the symmetric point in the normalised problem. This is true if and only if Cardinal Utility Symmetry holds. □

A.4 Proof of Lemma 14

Proof Consider any two normal Conditional Bargaining Problems $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$ and $\langle A', d', \succcurlyeq_1, \succcurlyeq_2, E\rangle$. Since preferences are Biseparable, positive affine transformations (two for each individual) of the utilities exist, denote them v_1, v_2 and w_1, w_2, with $v_1(M_1) = v_2(M_2) = w_1(M'_1) = w_2(M'_2) = 1$ and $v_1(d_1) = v_2(d_2) = w_1(d'_1) = w_2(d'_2) = 0$. Then, Subjective Contraction Independence holds if and only if the following implication holds: $w(A') \subseteq v(A)$, $S(\langle v(A), v(d)\rangle) = (\lambda^*_1, \lambda^*_2)$, and $(\lambda^*_1, \lambda^*_2) \in w(A')$, jointly imply $S(\langle w(A'), w(d')\rangle) = (\lambda^*_1, \lambda^*_2)$.

Consider any other utility representations of the two Conditional Bargaining Problems. If there is a non-empty set of quartets of positive affine transformations (each individual has one for each problem), each quartet transforming the problems to satisfy conditions i., ii. and iii. of Cardinal Contraction Independence, then those that normalise each problem *must* be a quartet in this set. When this is possible, all different utility representations of the two problems' solution alternatives must map to the same point, $(\lambda_1^*, \lambda_2^*)$, in the normalised problems. Equivalently, Cardinal Contraction Independence holds. □

A.5 Proof of Theorem 15

Suppose preferences are Biseparable. P1 is clearly equivalent to S1. For normal problems, Lemma 13 established the equivalence of P2 and S2*, and Lemma 14 established the equivalence of P3 and S3*. As discussed above, this immediately extends to non-normal problems. By Observation 12 the Conditional Nash Solution has the same utility representation as the Nash Solution, and a Conditional Bargaining Problem is compact and Subjective Mixture Closed if and only if it is compact and convex in utility space. Now apply Theorem 10.

A.6 Proof of Theorem 17

Let S be a Bargaining Solution satisfying axioms S1*, S2* and S3*. Let $\mathscr{K}$ denote the Kalai–Smorodinsky Solution. That $\mathscr{K}$ satisfies S1*, S2* and S3* is elementary. We already know $\mathscr{K}$ is well defined and satisfies S1*, S2, S3 and S4.

Take any a-Comprehensive Bargaining Problem, $\langle U, a\rangle$. We will show $S(\langle U, a\rangle) = \mathscr{K}(\langle U, a\rangle) = \lambda$. Define $\langle U', a'\rangle = \langle \gamma U + \delta, \gamma a + \delta\rangle$, where $\gamma_1, \gamma_2, \delta_1, \delta_2$ are chosen so that $B(\langle U', a'\rangle) = (B'_1, B'_2) = (1, 1)$ and $a' = (0, 0)$. This problem is a'-Comprehensive. Since $\mathscr{K}$ satisfies S4, $\mathscr{K}(\langle U', a'\rangle) = \lambda' = \gamma\lambda + \delta$. Now consider the following sets:

$$T := \text{comp}[\lambda'; a'] \setminus \{\lambda' + \mathbb{R}^2_{++}\}$$
$$T' := \text{comp}[(B'_1, a'_1), (a'_2, B_2), \lambda'; a']$$

The problems $\langle T, a'\rangle$ and $\langle T', a'\rangle$ are symmetric, so by S1* and S2*,

$$S(\langle T, a'\rangle) = S(\langle T', a'\rangle) = \lambda'$$

Notice that $T \supseteq U' \supseteq T'$, so using S3*:

$$S(\langle T, a'\rangle) \geqslant \gamma S(\langle U, a\rangle) + \delta \geqslant S(\langle T', a'\rangle)$$

Combining the above, $S(\langle U, a\rangle) = \mathscr{K}(\langle U, a\rangle)$ as required.

A.7 Proof of Lemma 20

Proof Consider any two normal Conditional Bargaining Problems $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$ and $\langle A', d', \succcurlyeq_1, \succcurlyeq_2, E\rangle$. Since preferences are Biseparable, there exists positive affine transformations (two for each individual) of the utilities involved, denote them v_1, v_2 and w_1, w_2, with $v_1(M_1) = v_2(M_2) = w_1(M'_1) = w_2(M'_2) = 1$ and $v_1(d_1) = v_2(d_2) = w_1(d'_1) = w_2(d'_2) = 0$. Then conditions *i.* and *ii.* of Subjective Monotonicity hold if and only if, for all $\alpha \in [0, 1]$:

$$\Phi_i(\alpha; v(A)) = \Phi_i(\alpha; w(A')) \text{ and } \Phi_j(\alpha; v(A)) \supseteq \Phi_j(\alpha; w(A'))$$

The solution S gives j a feasible outcome for each problem. Denote these y_j and y'_j. These outcomes are Subjective Mixtures of j's best and worst outcomes in each problem. Let λ and λ' denote the values of these Subjective Mixtures. The values of these mixtures coincide with the utility numbers in the normalised representation. So, let $v_j(y_j) = \lambda$ and $w_j(y'_j) = \lambda'$. Subjective Monotonicity requires that $\lambda \geqslant \lambda'$, or equivalently $v_j(y_j) \geqslant w_j(y'_j)$.

We chose normalised utilities v_j and w_j so, as a consequence, condition *ii.* of Subjective Monotonicity holds if and only if, for all $\alpha \in [0, 1]$, $\Phi_j(\alpha; v(A)) \supseteq \Phi_j(\alpha; w(A'))$. If we replace the utilities with any other positive affine transforms, $\gamma v_j + \delta$ and $\gamma' w_j + \delta'$, to maintain the equality $\gamma v_j(d_j) + \delta = \gamma' w_j(d_j) + \delta'$, we must have $\delta = \delta'$. Then condition *ii.* remains true if and only if $\gamma \geqslant \gamma'$. Therefore, whenever there are transformations that assign the same utility numbers to the disagreement points and preserve condition *ii.*, it follows that $\gamma v_j(y_j) + \delta \geqslant \gamma' w_j(y'_j) + \delta'$. This is true if and only if Cardinal Utility Monotonicity holds. □

A.8 Proof of Theorem 21

Suppose preferences are Biseparable. P1* is clearly equivalent to S1*. For normal problems, Lemma 13 established the equivalence of P2 and S2*, and Lemma 20 established the equivalence of P5 and S5*. As discussed, this immediately extends to non-normal problems. By Observation 19 the Conditional Kalai–Smorodinsky Solution has the same utility representation as the Kalai–Smorodinsky Solution, and a normal Conditional Bargaining Problem is compact and d-Comprehensive if and only if it is compact and $u(d)$-Comprehensive in utility space. Now apply Theorem 17.

A.9 Proof of Theorems 22 and 23

The proofs of Theorems 22 and 23 are outlined together. I detail the proof that, given Weak Pareto Efficiency, a solution satisfies Restricted Subjective Symmetry and Continuity (P2* and P6) only if satisfies Subjective Symmetry. Once the proof of this is understood, the remaining conditions may be verified similarly, so the details are omitted.

Fix a disagreement point, d, and consider a solution as a function $S : \mathscr{A} \to X_1 \times X_2$. Preferences are Biseparable with $u_i : X_i \to \mathbb{R}$, $i = 1, 2$, being the associated the utility functions. Define $u : X_1 \times X_2 \to \mathbb{R}^2$ such that $u = (u_1, u_2)$. Let $c(\mathbb{R}^2)$ be the set of compact subsets of $\mathbb{R}^2$, endowed with the Hausdorff metric. Define $\mathscr{U} : \mathscr{A} \to c(\mathbb{R}^2)$ so that for $A \in \mathscr{A}$, $\mathscr{U}(A) = u(A)$. Define $S^u : c(\mathbb{R}^2) \to \mathbb{R}^2$ such that $S^u = \mathscr{U} \circ S$.

For clarification, one may verify that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\mathscr{U}} & c(\mathbb{R}^2) \\ {\scriptstyle S}\big\downarrow & & \big\downarrow{\scriptstyle S^u} \\ X_1 \times X_2 & \xrightarrow[u]{} & \mathbb{R}^2 \end{array}$$

By the above construction, S^u is continuous in the Hausdorff metric if and only if S is continuous in the topology of closed convergence.

I now show that, given P1*, axioms P2* and P6 imply axiom P2. Let $\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle$ be a Conditional Bargaining Problem satisying condition *i.* of axiom P2*. Choose the utilities that normalise this problem, $u(M_1, M_2) = (1, 1)$ and $u(d_1, d_2) = (0, 0)$, and denote $\mathscr{U}(A) = T$. Condition *i* of P2* translates to the following condition for T: for any $\alpha, \beta \in \mathscr{D}$, $(\alpha, \beta) \in T$ iff $(\beta, \alpha) \in T$. That is, the set $T' := T \cap \mathscr{D}^2$ is symmetric. By the denseness of the dyadic rationals, the closure of T' equals T, so T is easily shown to be symmetric.

There is a *unique* point that is symmetric and efficient, denote this (λ, λ). I show that $S^u(T) = (\lambda, \lambda)$. Firstly, if $\lambda \in \mathscr{D}$, then condition *ii.* of P2* is satisfied and the result follows. Now suppose $\lambda \notin \mathscr{D}$. Clearly, λ can be approximated using dyadic rationals. That is, there exists a sequence $\{\alpha_k \lambda\}_{k\in\mathbb{N}}$ where, for each k, $\alpha_k\lambda \in \mathscr{D}$ and $\lim_{k\to\infty}\{\alpha_k\lambda\}_{k\in\mathbb{N}} = \lambda$. Now consider the sequence of sets $\{\alpha_k T\}_{k\in\mathbb{N}}$. For each k, $\alpha_k T$ is symmetric, with a unique symmetric and efficient point $(\alpha_k\lambda, \alpha_k\lambda)$. Furthermore, $\alpha_k\lambda \in \mathscr{D}$, so for each k, $S^u(\alpha_k T) = (\alpha_k\lambda, \alpha_k\lambda)$ by axiom P2*. Since $(\alpha_k\lambda, \alpha_k\lambda) \to (\lambda, \lambda)$, and S^u is continuous in the Hausdorff metric, $S^u(T) = (\lambda, \lambda)$. This holds if and only if $S(\langle A, d, \succcurlyeq_1, \succcurlyeq_2, E\rangle) = (\lambda M_1 \oplus_1 (1-\lambda)d_1, \lambda M_2 \oplus_2 (1-\lambda)d_2)$, and axiom P2 follows as required.

Using similar approaches to the above, one may show that the restricted versions of P3 and P5, along with Continuity P6, imply the full versions of P3 and P5. Finally, one can appeal to Theorems 15 and 21 to complete the proofs of Theorems 22 and 23, respectively.

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