Time Consistency and Decreasing Impatience.

Craig S. Webb

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2. The implications are characterised for a selection of known discounting models.
3. Some alternatives are presented.
4. A particular discount function is axiomatically characterised.

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- *Consumption streams* are $\mathcal{T}$-measurable functions $\mathbf{x} : T \to X$, the set of which is $\mathcal{C}$.
- For $0 \leqslant a < b < \infty$ and $\mathbf{z} \in \mathcal{C}$, denote by $\mathcal{C}(a, b, \mathbf{z})$ denote the set of consumption streams $\mathbf{x} \in \mathcal{C}$ such that $t \notin [a, b)$ implies $\mathbf{x}(t) = \mathbf{z}(t)$.

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$$V_r(\mathbf{x}) = \int_r^\infty D(t-r)u(\mathbf{x}(t))dt,$$

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with $u : X \to \mathbb{R}$ a utility function for outcomes, and $D : T \to \mathbb{R}$ a strictly decreasing and continuous discount function.

- *Regular* if $D(0) = 1$ and $\lim_{t \to \infty} D(t) = 0$.

Exponential Discounting.

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For all $0 \leqslant a \leqslant b < \infty$, $\mathbf{x}, \mathbf{y} \in \mathcal{C}(a, b, \mathbf{z})$ and $r, s \leqslant a$, we have $\mathbf{x} \succcurlyeq_r \mathbf{y}$ if and only if $\mathbf{x} \succcurlyeq_s \mathbf{y}$.

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Theorem 1.

Under regular discounted utility, time consistency holds if and only if exponential discounting holds.

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- Exponential discounting is a model of **constant impatience**.
- But, **decreasing impatience** (present bias, as a special case) seems to be prevalent.
- Many discount functions have been proposed to model this...

Discount functions for Decreasing Impatience.

1. **Generalised Hyperbolic:**

$$D(t) = (1 + \alpha t)^{-\frac{\beta}{\alpha}}, \qquad t \geqslant 0, \alpha \geqslant 0, \beta > 0.$$

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$$D(t) = \beta^{\alpha^{t}}, \qquad t \geqslant 0, \alpha \geqslant 0, \beta \in (0,1).$$

4. **Double Exponential:**

$$D(t) = \omega\delta^{t} + (1-\omega)\gamma^{t}, \qquad t \geqslant 0, \delta, \gamma, \omega \in (0,1).$$

An Example to Consider.

- Consider a set containing consumption streams of the following form:

$$\mathbf{x} = \begin{cases} \text{£}1200 & \text{if } t \in A \\ \text{£}1000 & \text{if } t \notin A \end{cases}$$

where A is a subset of the interval [27 months, 28 months).

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- This set contains streams that are:
  - Relatively easy to compare.
  - Differ only in a relatively short period, that is distant from the present.
- Suppose that, at time 0, the decision maker has complete preferences over this set.
- Is there a compelling reason for his preferences to change during, say, the next month?

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Local time consistency

Local time consistency holds on $\mathcal{C}(a, b, \mathbf{z})$ if there is an interval $[\underline{t}, \overline{t}]$ with $0 \leqslant \underline{t} < \overline{t} \leqslant a$ such that, for all $r, s \in [\underline{t}, \overline{t}]$ and all $\mathbf{x}, \mathbf{y} \in \mathcal{C}(a, b, \mathbf{z})$ we have $\mathbf{x} \succcurlyeq_r \mathbf{y}$ if and only if $\mathbf{x} \succcurlyeq_s \mathbf{y}$.

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- Is local time consistency reasonable to assume?
    - It is not falsifiable.
    - The set of acts $\mathcal{C}(a, b, \mathbf{z})$ is large, but much smaller than $\mathcal{C}$.
    - The time intervals $[\underline{t}, \overline{t}]$ and $[a, b)$ can be *arbitrarily* short.

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Under discounted utility, the following statements are equivalent:

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- Non-exponential discount functions **must** violate time consistency.
- But, are the above functions **too inconsistent?**

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- Discount functions with exponential pieces are an alternative.

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- **E.g. Continuous Quasi-Hyperbolic:**

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with $\delta_S, \delta_L \in (0,1)$, $\beta = (\frac{\delta_S}{\delta_L})^S$, and $S \geqslant 0$.

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- Recent evidence suggests increasing and decreasing impatience are common.
- Sayman and Onculer (2008), *reversed-S* pattern.

Increasing and Decreasing Impatience.

- Consider the following discount function:

$$D(t) = \begin{cases} \delta_S^t & \text{if } 0 \leqslant t < S, \\ \beta\delta_M^t & \text{if } S \leqslant t < M, \\ \beta\gamma\delta_L^t & \text{if } M \leqslant t < \infty, \end{cases}$$

where $\delta_S, \delta_M, \delta_L \in (0,1)$ and $\beta = \left(\frac{\delta_S}{\delta_M}\right)^S$ and $\gamma = \beta\left(\frac{\delta_M}{\delta_L}\right)^M$.

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  - $[0, S]$ is the **short term.**
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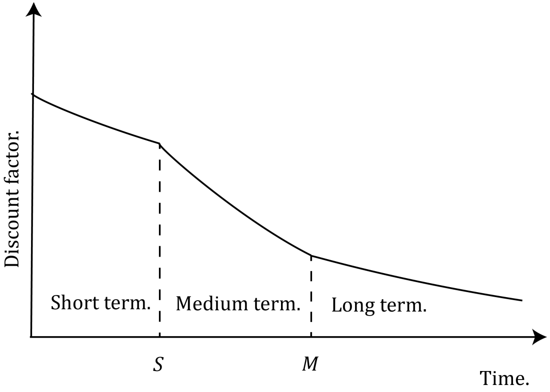
- In this representation:
    - $[0, S]$ is the **short term.**
    - $[S, M]$ is the **medium term.**
    - $[M, \infty)$ is the **long term.**
- Note that this stratification is **subjective** —S and M are **personal parameters**.

The Reversed-S Hypothesis.



Proposition

The reversed-S hypothesis $(\delta_M < \delta_S < \delta_L)$ *holds iff* $\beta > 1$, $\gamma < 1$ *and* $\beta\gamma < 1$.

Restricted Time Consistency Conditions.

Time-consistency-before-σ-from-now.

For all $0 \leqslant a \leqslant b < \infty$, $\mathbf{x}, \mathbf{y} \in \mathcal{C}(a, b, \mathbf{z})$ and $b - r \leqslant \sigma$ and $b - s \leqslant \sigma$, we have $\mathbf{x} \succcurlyeq_r \mathbf{y}$ if and only if $\mathbf{x} \succcurlyeq_s \mathbf{y}$.

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Time-consistency-beyond-τ-from-now.

For all $0 \leqslant a \leqslant b < \infty$, $\mathbf{x}, \mathbf{y} \in \mathcal{C}(a, b, \mathbf{z})$ and $\tau \leqslant a - r$ and $\tau \leqslant a - s$, we have $\mathbf{x} \succcurlyeq_r \mathbf{y}$ if and only if $\mathbf{x} \succcurlyeq_0 \mathbf{y}$.

Restricted Time Consistency Conditions.

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- For appropriate σ and τ — consistency when making medium-term decisions.
- The above conditions are not axioms — for what σ and τ should they hold?
- Key idea is a simplification assumption —*all delays can be classified as short, medium or long term*.

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Axiom: Three-Stage Time Consistency.

For **all** $0 \leqslant \sigma \leqslant \tau < \infty$, **at least one** of the following holds:

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Characterisation.

Theorem

Under regular discounted utility, the following statements are equivalent:

Continuous quasi-hyperbolic discounting holds if $(\delta_S - \delta_M)(\delta_M - \delta_L) = 0$, and exponential discounting holds if $\delta_S = \delta_M = \delta_L$.

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1. *Three-stage time consistency holds.*
2. *There exists $S, M \in T$ and constants $\delta_S, \delta_M, \delta_L \in (0,1)$ such that:*

$$D(t) = \begin{cases} \delta_S^t & \text{if } 0 \leqslant t < S, \\ \beta\delta_M^t & \text{if } S \leqslant t < M, \\ \beta\gamma\delta_L^t & \text{if } M \leqslant t < \infty, \end{cases}$$

where $\beta = \left(\frac{\delta_S}{\delta_M}\right)^S$ and $\gamma = \beta\left(\frac{\delta_M}{\delta_L}\right)^M$. The parameters are uniquely defined when meaningful.

Continuous quasi-hyperbolic discounting holds if $(\delta_S - \delta_M)(\delta_M - \delta_L) = 0$, and exponential discounting holds if $\delta_S = \delta_M = \delta_L$.

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- Thanks.