Piecewise Linear Rank-Dependent Utility.

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August 10, 2016

Abstract

Choice under risk is modelled using a piecewise linear version of rank-dependent utility. This model can be considered a continuous version of NEO-expected utility (Chateauneuf, Eichberger and Grant, 2007). In a framework of objective probabilities a preference foundation is given, without requiring a rich structure on the outcome set. The key axiom is called *complementary additivity*.

Keywords: Rank-dependent utility; security and potential level preferences; optimism; pessimism.

JEL Classification: D81.

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1 Introduction

When considering choice under risk, evidence suggests that most decision makers are simultaneously pessimistic and optimistic — they are *ambivalent*. It has been argued before that these departures from expected utility can be explained by taking into account the particular salience of the best and worst outcomes of decisions (Lopes, 1987; Cohen, 1992). An additional focus on the worst outcome is akin to pessimism, and on the best outcome is akin to optimism. In this way, the *NEO-expected utility* model (Chateauneuf et al. 2007) elegantly extends expected utility to incorporate ambivalence.

NEO-expected utility successfully organises several robust empirical findings of choice under risk. It allows for optimism and pessimism, in the sense of Wakker (1994), but also retains expected utility "inside the probability triangle", where violations are less frequently observed (Abdellaoui and Munier, 1998). Due in most part to its tractable form, NEO-expected utility has been applied extensively. [1] Departures from expected utility are captured using discontinuities in the evaluation formula. Because of these discontinuities, however, the axiomatic foundations of NEO-expected utility are much more complicated than expected utility (see Webb and Zank, 2011).

Piecewise linear rank-dependent utility (RDU_{PL}), the model considered in this paper, is a continuous version of NEO-expected utility. RDU_{PL} is the special case

[1] NEO-expected utility has been used by Abdellaoui et al (2010), Dominiak et al. (2012), Dominiak and Lefort (2013), Ford et al. (2013), Eichberger et al. (2012), Eichberger and Kelsey, (2011), Eichberger and Kelsey, (2014), Ludwig and Zimper (2014), Romm (2014), Teitelbaum (2007), and Zimper (2012).

of rank-dependent utility with a piecewise linear probability weighting function. The well known “inverse-S shaped” probability weighting scheme associated with ambivalence is approximated under RDU_{PL} with “stretched-N” shaped probability weighting. RDU_{PL} could be called an *empirical generalisation* of NEO-expected utility. In terms of observable choices, NEO-expected utility cannot be distinguished from RDU_{PL}. In this sense, the foundational difficulties of NEO-expected utility are resolved with very little cost. Furthermore, RDU_{PL} allows for some additional realism. For example, optimism and pessimism have been observed for non-extreme outcomes (see, for example, Wu and Gonzalez, 1996), which is captured to some extent by RDU_{PL} in a way that is ruled out by NEO-expected utility.

Webb (2015) gave an axiomatisation of the analogue of the RDU_{PL} model under purely subjective uncertainty — the Savage framework. In this paper, RDU_{PL} is axiomatised using objective probabilities — the von Neumann-Morgenstern framework. Only the richness of the probability interval is used, the outcome set can be arbitrary. Hence, the theory may be applied to monetary outcomes, health outcomes, indivisible goods, and so on. The key axiom under risk, called *complementary additivity*, is more intuitive, easier to test empirically, and the proof is shorter and more direct. The formal definitions are outlined in Section 2. The piecewise linear rank-dependent utility model is presented in Section 3. A simple tradeoff axiom, necessary for rank-dependent utility, is presented in Section 4. The axiomatic foundation of piecewise linear rank-dependent utility presented in Section 5. All proofs are contained in the Appendix.

2 Choice Under Risk

There is a set of outcomes $X = \{x_0, \dots, x_n\}$, with $n \geqslant 2$, and a strict order over outcomes, such that $x_0 \prec \cdots \prec x_n$.[2] The results of this paper are more elegantly presented describing lotteries in their decumulative form, as in Abdellaoui (2002) and others (Diecidue, Schmidt and Zank, 2009; Webb and Zank, 2011). We write a lottery $p = (p_1, \dots, p_n)$ where $1 \geqslant p_1 \geqslant \cdots \geqslant p_n \geqslant 0$. A coordinate p_i of a lottery p denotes the *decumulative* probability of outcome x_i, that is, the probability (in the standard sense) of receiving an outcome x_i *or better*. The decumulative probability of x_0 equals one in all lotteries, hence we exclude it from the notation. The set of lotteries is $\mathcal{L}_X = \{p = (p_1, \dots, p_n) \in \mathbb{R}^n : 1 \geqslant p_1 \geqslant \cdots \geqslant p_n \geqslant 0\}$. We consider lotteries on a finite X for ease of exposition. The results of this paper can be extended to lotteries with finite support on an infinite outcome set, as done in, e.g., Abdellaoui (2002).

Preferences $\succsim$ are defined over $\mathcal{L}_X$. Preferences over degenerate lotteries agree with the strict order on outcomes. The set $\mathcal{L}_X$ is a mixture space, with mixtures of lotteries defined point-wise. For $p, q \in \mathcal{L}_X$ and $\alpha \in (0, 1)$, $r = \alpha p + (1 - \alpha) q \in \mathcal{L}_X$ is such that $r_i = \alpha p_i + (1 - \alpha) q_i$ for all $i = 1, \dots, n$. Given $p \in \mathcal{L}_X$ and $\rho \in [0, 1]$, the notation $p + \rho_i$ denotes the lottery $(p_1, \dots, p_{i-1}, p_i + \rho, p_{i+1}, \dots, p_n)$. It is implicit in this notation that $p_{i-1} \geqslant p_i + \rho \geqslant p_{i+1}$.

Preferences $\succsim$ are represented by a real-valued utility function $U : \mathcal{L}_X \to \mathbb{R}$ if the following equivalence holds: $p \succsim q \Leftrightarrow U(p) \geqslant U(q)$. If a utility representation exists, preferences $\succsim$ are a weak order, that is, they are complete and transitive.

[2] A weak order over outcomes can be assumed instead of a strict order. In that case it is required that X has at least three indifference classes when passing to the quotient.

Monotonicity holds if $p_i \geqslant q_i$ for all $i = 1, \ldots, n$ and $p \neq q$ implies $p \succ q$. That is, monotonicity requires that preferences respect first-order stochastic dominance. *Jensen continuity* holds if, for all $p, q, r \in \mathcal{L}_X$ with $p \succ q$ there exists $\alpha, \beta \in (0, 1)$ such that $\alpha p + (1 - \alpha) r \succ q$ and $p \succ \beta q + (1 - \beta) r$. *Continuity* holds if, for all $p \in \mathcal{L}_X$, the sets $\{q \in \mathcal{L}_X : q \succ p\}$ and $\{q \in \mathcal{L}_X : q \prec p\}$ are open in $\mathcal{L}_X$. For monotonic, weak ordered preferences, Jensen continuity holds only if continuity holds (see Lemma 18 of Abdellaoui, 2002).

3 Piecewise Linear Rank-Dependent Utility

Expected utility holds if there is a utility for outcomes $u : X \to \mathbb{R}$ such that the map,

$$p \mapsto \mathrm{EU}(p) = u(x_0) + \sum_{j=1}^{n} p_j [u(x_j) - u(x_{j-1})]$$

represents preferences $\succsim$ over $\mathcal{L}_X$. Probabilities enter into the expected utility formula linearly. Such preferences, therefore, necessarily satisfy the following axiom:

Axiom (Additivity): For all $p, q, p + r_i, q + r_i \in \mathcal{L}_X$: $p \succsim q \Leftrightarrow p + r_i \succsim q + r_i$.

In the presence of other basic axioms, additivity is sufficient for expected utility (see Theorem 5 of Webb and Zank, 2011). *Rank-dependent utility* (Quiggin, 1982) holds if there is a utility for outcomes $u : X \to \mathbb{R}$ and a strictly increasing probability weighting function $w : [0, 1] \to [0, 1]$, with $w(0) = 0$ and $w(1) = 1$, such that the map,

$$p \mapsto \mathrm{RDU}(p) = u(x_0) + \sum_{j=1}^{n} w(p_j) [u(x_j) - u(x_{j-1})]$$

represents preferences $\succsim$ over $\mathcal{L}_X$. Expected utility is the special case of rank-dependent utility, with w the identity function. For a lottery p, let m_p denote the worst possible outcome and M_p denote the best possible outcome. *Security / potential level preferences* (Cohen, 1992) are represented by the following map:

$$p \mapsto \mathrm{SP}(p) = f(m_p, M_p)\mathrm{EU}(p) + g(m_p, M_p)$$

where $f : X \times X \to \mathbb{R}$ and $g : X \times X \to \mathbb{R}$ are real-valued functions.

A popular model of choice under risk is *NEO-expected utility* (Chateaneuf, Eichberger and Grant, 2007). NEO-expected utility holds if there is a utility for outcomes $u : X \to \mathbb{R}$ and parameters $\gamma, \delta \geqslant 0$, with $\gamma + \delta < 1$, such that the map,

$$p \mapsto \mathrm{NEO}(p) = \gamma u(m_p) + (1 - \gamma - \delta)\mathrm{EU}(p) + \delta u(M_p)$$

represents preferences $\succsim$ over $\mathcal{L}_X$. NEO-expected utility unites several popular models of choice under risk. It is the special case of rank-dependent utility, with the following probability weighting function:

$$w_{\mathrm{NEO}}(p) = \begin{cases} 0 & \text{if } p = 0, \\ (1 - \gamma - \delta)p + \delta & \text{if } 0 < p < 1, \\ 1 & \text{if } p = 1, \end{cases}$$

with $\gamma, \delta \geqslant 0$ and $\gamma + \delta < 1$. NEO-expected utility also coincides with Cohen's security / potential level model when, for all $x, y \in X$, $f(x, y) = 1 - \gamma - \delta$ and $g(x, y) = \gamma u(x) + \delta u(y)$ with $\gamma, \delta \geqslant 0$ and $\gamma + \delta < 1$.

NEO-expected utility incorporates optimism and pessimism into expected utility in a tractable way. Departures from expected utility are captured using discontinuities in the evaluation formula. Because of these discontinuities, however, the axiomatic foundations of NEO-expected utility are much more complicated than expected utility (see Webb and Zank, 2011). However, notice that one cannot empirically distinguish between w_{NEO} and any probability weighting w that is continuous on $[0, 1]$ and linear on an interval $[\kappa, 1-\kappa]$, so long as κ is close enough to zero. To see this, suppose that, for probability values $p_1 < \cdots < p_n$, a finite data set of probability weighting function values have been obtained: $\{w(p_1), \ldots, w(p_n)\}$. Suppose, also, that NEO-expected utility cannot be rejected, so that there is $\gamma, \delta \geqslant 0$ with $\gamma + \delta < 1$ such that $\{w(p_1), \ldots, w(p_n)\} = \{(1-\gamma-\delta)p_1 + \delta, \ldots, (1-\gamma-\delta)p_n + \delta\}$. Let $\kappa = \min\{p_1, 1-p_n\}$. Then, one cannot reject the following functional form:

$$w(p) = \begin{cases} \phi(p) & \text{if } p < (1-\kappa), \\ (1-\gamma-\delta)p + \delta & \text{if } (1-\kappa) \leqslant p \leqslant \kappa, \\ \psi(p) & \text{if } \kappa < p \leqslant 1, \end{cases}$$

where ϕ and ψ are functions chosen so that w is increasing and continuous on $[0, 1]$. Webb (2015), motivated by tractability considerations, suggested assuming linearity also on $[0, \kappa]$ and $[1-\kappa, 1]$. The resulting continuous probability weighting function with such properties is w_{PL} given by:

$$w_{\mathrm{PL}}(p) = \begin{cases} (1/\kappa)(\kappa(1-\gamma) + (1-\kappa)\delta)p & \text{if } 0 \leqslant p < \kappa, \\ (1-\gamma-\delta)p + \delta & \text{if } \kappa \leqslant p \leqslant (1-\kappa), \\ (1/\kappa)((1-(\kappa\delta + (1-\kappa)(1-\gamma))p - (1-\kappa)(\gamma+\delta) + \delta)) & \text{if } (1-\kappa) < p \leqslant 1. \end{cases}$$

with $0 \leqslant \kappa \leqslant \frac{1}{2}$, $\kappa > \frac{-\delta}{1-\gamma-\delta}$ and $\kappa > \frac{-\gamma}{1-\gamma-\delta}$ and $\gamma + \delta < 1$.

Ambivalence, exhibiting both optimism and pessimism, is captured by w_{PL} if $\gamma > 0$, $\delta > 0$ and $\kappa < \frac{1}{2}$. In general, it is not required that γ and δ are non-negative. In principle, w_{PL} can be very steep in the middle region, but remains everywhere strictly increasing and normalised at the extremes. *Piecewise linear rank-dependent utility*, denoted $\mathrm{RDU}_{\mathrm{PL}}$, is the special case of rank-dependent utility with probability weighting function w^{PL}. Expected utility holds as the special case of $\mathrm{RDU}_{\mathrm{PL}}$ with $\gamma = \delta = 0$.

4 Elementary Tradeoff Consistency

For lotteries involving only two outcomes, there are no observable differences between expected utility and rank-dependent utility.[3] The simplest objects for the study of probabilistic risk attitudes are, therefore, three-outcome lotteries. Let $Y \subseteq X$ be a set of three outcomes. Relabel the outcomes of Y as $x_{(0)} \prec x_{(1)} \prec x_{(2)}$ and let $\mathcal{L}_Y$ denote the set of lotteries over Y. In decumulative notation, lotteries in $\mathcal{L}_Y$ can be written $(p_{(1)}, p_{(2)})$, $(q_{(1)}, q_{(2)})$, and so on.

Suppose that a decision maker expresses the indifference $(a, p) \sim (b, q)$. One may interpret this indifference to mean that the impact of replacing a with b, in the $p_{(1)}$-coordinate, is exactly equal to the impact of replacing p with q in the $p_{(2)}$-coordinate. Now suppose that a and b are replaced by c and d, respectively, and the decision maker remains indifferent, $(c, p) \sim (d, q)$. Then, it seems, the impact of replacing c with d, in the $p_{(1)}$-coordinate, is also equal to the impact of replacing p with q in the $p_{(2)}$-coordinate. As Abdellaoui (2002) puts it, using p and q in the $p_{(2)}$-coordinate as our "measuring rod", the replacements of a with b and c with d are equivalent *probability tradeoffs*.

[3] The order induced by the first-order stochastic dominance relation is complete in the two-outcome case.

Probability tradeoffs are a useful tool for analysing probabilistic risk attitudes. For example, the *common consequence effect* (Allais, 1952) suggests that the probability tradeoff 11% to 10% seems to have less impact than the probability tradeoff from certainty to 99%. For such claims to be clear and meaningful, however, the notion of probability tradeoffs having greater, less, or the same impact must be independent of the particular "measuring rod" and coordinate used. Suppose that one observed $(r, a) \sim (s, b)$ but $(r, c) \nsim (s, d)$. Now, r and s in the $p_{(1)}$-coordinate are the "measuring rod", and the probability tradeoffs occur in the $p_{(2)}$-coordinate. The initial indifferences above were interpreted to mean $a : b$ probability tradeoff is equivalent to that of a $c : d$ probability tradeoff, but a different conclusion would be reached in this case. The following axiom rules out such inconsistency:

Axiom (Elementary Tradeoff Consistency): For all $Y \subseteq X$, with $|Y| = 3$, and lotteries $(a, p), (b, q), (c, p), (d, q), (r, a), (s, b), (r, c), (s, d) \in \mathcal{L}_Y$, if $(a, p) \sim (b, q)$ and $(c, p) \sim (d, q)$ and $(r, a) \sim (s, b)$, then $(r, c) \sim (s, d)$.

The adjective "elementary" refers to the fact that the axiom applies only to lotteries involving three outcomes, and also that the axiom uses only indifferences. If $|X| = 3$, then elementary tradeoff consistency is equivalent to the *probability tradeoff consistency* axiom of Abdellaoui (2002), except that it is formulated only for indifferences (Köbberling and Wakker, 2003).

Under rank-dependent utility, the impact of probability tradeoffs are characterised entirely by the probability weighting function. Assuming rank-dependent utility with probability weighting function w, the indifferences $(a, p) \sim (b, q)$ and $(c, p) \sim (d, q)$ hold simultaneously if and only if:

$$w(a) - w(b) = w(c) - w(d).$$

Neither the “measuring rods”, p and q, nor any suggestion of the tradeoff coordinate used, appear in the above equation. Therefore, under rank-dependent utility, these can be replaced, and $(r, a) \sim (s, b)$ holds if and only if $(r, c) \sim (s, d)$ holds.

Observation 1. *Rank-dependent utility (hence, also piecewise linear rank-dependent utility) holds only if preferences $\succsim$ on $\mathcal{L}_X$ satisfy elementary tradeoff consistency.*

In Appendix A, elementary tradeoff consistency is used to provide a new preference foundation for rank-dependent utility.

5 Complementary Additivity

The security / potential level preferences model, of which NEO-expected utility is a special case, captures the behaviour of a decision maker who, when considering a lottery, classifies its outcomes into three classes: *certain*, *impossible*, and *risky*. That is, respectively, outcomes with decumulative probability one (receiving at least that amount is certain), outcomes with decumulative probability zero (impossible to receive better outcomes), and outcomes with decumulative probability between zero and one (receiving a better or a worse outcome is possible). This corresponds to identifying the very worst outcome, the very best outcome, and the non-extreme outcomes of a lottery. In those models, expected utility holds *within*, but not necessarily *across* the classes.

It has been noted by Abdellaoui, l’Haridon and Zank (2010: 51) that assuming the effects of optimism and pessimism are captured entirely by additional focus on the very best and worst outcomes is somewhat restrictive. Optimism and pessimism have been observed, to a lesser degree, for non-extreme outcomes (Wu and Gonzalez, 1996). The theory developed here proposes a similar three-criteria distinction, but extends the notions of best and worst to include some non-extreme outcomes. Consider a decision maker who, when considering a lottery, does the following:

1. Groups outcomes into three classes: *likely*, *unlikely*, and *moderate*. That is, respectively, outcomes with decumulative probability sufficiently (subjectively) high, outcomes with decumulative probability sufficiently low, and outcomes with decumulative probability between the first two classes. This stratification corresponds, essentially to the (*subjectively defined*) worst ranked outcomes, the best ranked outcomes, and the outcomes ranked not so extreme in any particular lottery.

2. Considers *likely* and *unlikely* to be *duals*. That is, an outcome with decumulative probability less than than α is considered "unlikely" if and only if outcomes with decumulative probability greater than $(1 - \alpha)$ are considered "likely".

The first assumption above relates to the mental processes used by decision makers when considering lotteries, which falls under the realm of cognitive psychology rather than utility theory. In this vein, Lopes (1987; 1996) has gathered a range of evidence supporting the view that the worst outcomes, best outcomes, and moderate outcomes (in particular, in that order) seem to be especially salient when decision makers think about lotteries. The second assumption is defended more on intuitive grounds. For example, if a decision maker refers to probabilities less than one third as "unlikely", it would seem, linguistically at least, that the remaining probability (larger than two thirds) is sufficiently large to be considered "likely". That is, we suppose that saying "receiving this outcome or better is unlikely" means the same thing as "receiving this outcome or worse is likely".

In order to address the question of whether assumptions 1 and 2 are sufficiently realistic, overly simple, or just plain false, we require choice-based, testable implications. We now consider what behavioural implications may be reasonable for decision makers using this three-criteria process. The idea pursued here is that, when comparing lotteries, changes that affect these lotteries *without* affecting the subjective, three part structure of the outcomes are handled "rationally". That is, as the additivity axiom of expected utility suggests.[4] Given $\alpha \in [0, 1]$, consider the following condition:

Definition (α-Upper Additivity): For all $p, q, p + \rho_i, q + \rho_i \in \mathcal{L}_X$ with $p_i, q_i, p_i + \rho, q_i + \rho \geqslant \alpha$: $p \gtrsim q \Leftrightarrow p + \rho_i \gtrsim q + \rho_i$.

If α is large enough, so x_i is considered likely in p and q, then the additions above preserve the structure of p and q's three outcome classes. The decision maker then finds the comparison of $p + r_i$ and $q + r_i$ similar to the comparison of p and q, and does not reverse his preferences. If α were sufficiently small, so that x_i is considered unlikely in p and q, then the following condition seems more reasonable:

Definition (α-Lower Additivity): For all $p, q, p + \rho_i, q + \rho_i \in \mathcal{L}_X$ with $p_i, q_i, p_i + \rho, q_i + \rho \leqslant \alpha$: $p \gtrsim q \Leftrightarrow p + \rho_i \gtrsim q + \rho_i$.

These conditions can also be used to capture the decision maker considering likely and unlikely as duals:

Definition (α-Outer Additivity): α-lower additivity and $(1 - \alpha)$-upper additivity hold.

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[4] A similar notion motivates, under uncertainty, the *comonotonic independence* axiom, where act mixtures that preserve outcome ranking structure of the acts are handled "rationally".

The following condition captures the same intuition, considering changes affecting outcomes that are considered moderate (neither likely nor unlikely). By the assumed dual relationship if α is in the moderate region, then so is $(1-\alpha)$:

Definition (α-Inner Additivity): For all $p, q, p{+}\rho_i, q{+}\rho_i \in \mathcal{L}_X$ with $\alpha \leqslant p_i, q_i, p_i + \rho, q_i + \rho \leqslant 1-\alpha$: $p \succsim q \Leftrightarrow p+\rho_i \succsim q+\rho_i$.

Note that, the definition of α-outer additivity captures our initial intuition only if $0 \leqslant \alpha \leqslant \frac{1}{2}$, and the definition of α-inner additivity requires the same condition to hold. An axiom is now formulated that holds for all $0 \leqslant \alpha \leqslant \frac{1}{2}$. Because the distinction between "likely", "unlikely" and "moderate" is subjective, we may not know *a priori,* for a *specific* $0 \leqslant \alpha \leqslant \frac{1}{2}$, which of the above conditions is relevant. There is, however, a *complementarity* between these conditions that can be exploited. If assumptions 1 and 2 above are taken seriously, then, whenever α is small enough, we expect α-outer additivity to hold. Otherwise, we expect α-inner additivity to hold. Hence, when asked, "for which α does α-outer additivity hold?" our answer is, "those α for which α-inner additivity does not hold." Conversely, when asked, "for which α does α-inner additivity hold?" our answer is, "those α for which α-outer additivity does not hold." That is, the combination of assumptions 1 and 2 with the "additivity within classes" idea, leads to the following axiom:

Axiom (Complementary Additivity): For all $0 \leqslant \alpha \leqslant \frac{1}{2}$, preferences $\succsim$ either satisfy α-outer additivity, α-inner additivity, or both.

By substituting a RDU_{PL}, it can be established that RDU_{PL} necessarily satisfies both κ-outer additivity and κ-inner additivity. It follows that RDU_{PL} satisfies α-outer additivity for all $\alpha \leqslant \kappa$, and satisfies α-inner additivity for all $\kappa \leqslant \alpha \leqslant \frac{1}{2}$. Hence, complementary additivity is a necessary condition for RDU_{PL}. We can now state our main theorem:

Theorem 2. *Let $X = \{x_0, \dots, x_n\}$ with $n \geqslant 2$ and $x_0 \prec \cdots \prec x_n$. The preference relation $\succsim$ on $\mathcal{L}_X$ is a Jensen continuous, monotonic, weak order that satisfies elementary tradeoff consistency and complementary additivity if and only if piecewise linear rank-dependent utility holds. The utility function is unique up to positive affine transformation and the probability weighting function is unique.*

Theorem 2 is proved in Appendix B. It should also be noted that complementary additivity, even if combined with weak ordering, Jensen continuity, and monotonicity, does not imply elementary tradeoff consistency. Hence, the axioms used in Theorem 2 are independent. To establish this fact, consider the following example:

Example 3. *Let $X = \{x_0, \dots, x_n\}$ with $n \geqslant 2$ and $x_0 \prec \cdots \prec x_n$, and let preferences $\succsim$ over $\mathcal{L}_X$ be represented by the map:*

$$(p_1, \dots, p_n) \mapsto u(x_0) + \sum_{i=1}^{n} \phi_i(p_i)[u(x_i) - u(x_{i-1})]$$

with $u(x_0) < \cdots < u(x_n)$ and:

$$\phi_k(p_k) = \begin{cases} \frac{1}{2k} p_k & \text{if } 0 \leqslant p_k < \frac{1}{2}, \\ \frac{1}{2k} p_k + (1 - \frac{1}{2k})(2p_k - 1) & \text{if } \frac{1}{2} \leqslant p_k \leqslant 1. \end{cases} \quad \text{for all } k \in \{1, \dots, n\}.$$

Such preferences are necessarily continuous, monotonic, and weak ordered. Complementary additivity is necessarily satisfied. Elementary tradeoff consistency, however, is violated. Let $Y = \{x_0, x_i, x_j\}$, with $i < j$ and (a, p), (b, q), (c, p), (d, q), (r, a), (s, b), (r, c), $(s, d) \in \mathcal{L}_Y$ with $a, b < \frac{1}{2} < c, d$. Then, $(a, p) \sim (b, q)$ and $(c, p) \sim (d, q)$ are jointly equivalent to $a - b = (4i - 1)(c - d)$, whereas

$(r, a) \sim (s, b)$ and $(r, c) \sim (s, d)$ are jointly equivalent to $a - b = (4j - 1)(c - d)$. These cannot both hold, hence probability tradeoffs on the p_i-coordinate do not agree with those on the p_j-coordinate. As such, these preferences cannot be represented by rank-dependent utility, hence cannot be represented by piecewise linear rank-dependent utility.

One might conjecture that elementary tradeoff consistency can be dropped from Theorem 2 when the outcome set contains more than three outcomes. Example 3 proves that this is not the case. If there are more than three outcomes, then it is possible to derive an additive representation without appealing to elementary tradeoff consistency.[5] However, probability weighting cannot be separated from utility unless one further assumes elementary tradeoff consistency.

6 Closing Comments

In this paper we have considered a piecewise linear version of rank-dependent utility for choice under risk. The model can be considered a continuous version of NEO-expected utility. Empirically, evidence that fails to falsify NEO-expected necessarily fails to falsify piecewise linear rank-dependent utility. In terms of obtaining such evidence, the complementary additivity axiom presented here is the critical test. The theorem presented here is perhaps the simplest generalisation of the von-Neumann and Morgenstern theorem that accounts for ambivalent behaviour.

[5] In the appendix it is shown that complementary additivity implies a condition called *coordinate independence*, which is sufficient, in the presence of the basic axioms, to derive an additive representation.

Acknowledgements

I am grateful for the comments of an anonymous reviewer. The usual disclaimer applies.

Appendices

Appendix A: A Characterisation of Rank-Dependent Utility

For a lottery $p \in \mathcal{L}_X$ and $i \in \{1, \ldots, n\}$, we use the notation $r_i p = (t_1, \ldots, t_n)$ to denote the lottery with $t_j = r_j$ if $j = i$ and $t_j = p_j$ if $j \neq i$. It is implicit in this notation that $p_{i-1}, q_{i-1} \geqslant r_i \geqslant p_{i+1}, q_{i+1}$. *Coordinate independence* holds if, for all $p, q, r, s \in \mathcal{L}_X$, $r_i p \succsim r_i q$ if and only if $s_i p \succsim s_i q$. The following theorem characterises rank-dependent utility:

Theorem 4. *Let $X = \{x_0, \ldots, x_n\}$ with $n \geqslant 2$ and $x_0 \prec \cdots \prec x_n$. The preference relation $\succsim$ on $\mathcal{L}_X$ is a Jensen continuous, monotonic, weak order that satisfies elementary tradeoff consistency and coordinate independence if and only if rank-dependent utility holds. The utility function is unique up to positive affine transformation and the probability weighting function is unique.*

If there are precisely three outcomes ($n = 2$), then coordinate independence can be dropped from the axiom set in Theorem 4 and rank-dependent utility still holds.

Proof: The necessity of the axioms for the representation is routinely demonstrated by substituting the representation. We now establish the sufficiency of the axioms for rank-dependent utility. Lemma 18 of Abdellaoui (2002) establishes that continuity on $\mathcal{L}_X$ follows from weak ordering, monotonicity and Jensen continuity on $\mathcal{L}_X$. For $Y \subseteq X$ with $|Y| = 3$, elementary tradeoff consistency on $\mathcal{L}_Y$ is identical to the probability tradeoff consistency axiom of Abdellaoui (2002), except that it holds only for indifferences (Köbberling and Wakker, 2003). This is sufficient for rank-dependent utility to hold on each $\mathcal{L}_Y$ with $Y \subseteq X$ and $|Y| = 3$.

If $|X| = 3$, the proof is complete. Assume that $|X| > 3$. Theorem 3.2 of Wakker (1993) implies there exist extended real-valued, continuous, strictly monotone functions $V_1, \dots, V_n$ on $\mathcal{L}_X$ such that $\succsim$ is represented by the map $p \mapsto \sum_{j=1}^{n} V_j(p_j)$. The additive value functions $V_1, \dots, V_n$ are finite valued, except possibly V_1 at zero and V_n at one. Additive value functions $V_1, \dots, V_n$, for $n \geqslant 2$, are *jointly cardinal*, that is, if $\tilde{V}_1, \dots, \tilde{V}_n$ are also additive value functions representing $\succcurlyeq$ over $\mathcal{L}_X$, then $\tilde{V}_j = aV_j + b_j$ with $a > 0$ and $b_j \in \mathbb{R}$ for all $j = 1, \dots, n$.

For all $i, j \in \{1, \dots, n\}$, with $i \neq j$, preferences on $L_{\{x_0, x_i, x_j\}}$ are represented by $V_i + V_j$ and also by rank-dependent utility. Therefore, V_i is proportional to V_j. The additive value functions $V_1, \dots, V_n$ are therefore proportional to each other, hence proportional to their sum $\sum_{j=1}^{n} V_j$. These functions are therefore finite-valued at the extremes (see Proposition 3.5 of Wakker, 1993). Rescale the additive value functions so that $V_1(0) = \cdots = V_n(0) = 0$ and $\sum_{j=1}^{n} V_j(1) = 1$. Define a probability weighting function $w \equiv \sum_{j=1}^{n} V_j$. Then, preferences are represented by the map:

$$p \mapsto w(p_1)s_1 + \cdots + w(p_n)s_n \quad \forall p \in \mathcal{L}_X.$$

where $s_1, \dots, s_n > 0$. Define a utility function such that $u(x_0) = 0$ and $u(x_j) = u(x_{j-1}) + s_j$ for $j = 1, \dots, n$, and rank-dependent utility holds. ■

Appendix B: Proof of Theorem 2

Proof: The necessity of the axioms for the representation is routinely demonstrated by substituting the representation. Assume that preferences $\succsim$ over $\mathcal{L}_X$ are a continuous, monotonic weak order satisfying elementary tradeoff consistency and complementary additivity. We will derive a piecewise linear rank dependent utility representation. First, consider the implications of the complementary additivity axiom. Under weak ordering, if α-outer additivity holds for some $0 \leqslant \alpha \leqslant \frac{1}{2}$, then $\tilde{\alpha}$-outer additivity holds for all $\tilde{\alpha} \in [0, \alpha]$. Similarly, if α-inner additivity holds for some $0 \leqslant \alpha \leqslant \frac{1}{2}$, then $\tilde{\alpha}$-inner additivity holds for all $\tilde{\alpha} \in [\alpha, \frac{1}{2}]$. Let α^* be the largest value in $[0, \frac{1}{2}]$ such that α^*-outer additivity holds. Let α_* be the smallest value in $[0, \frac{1}{2}]$ such that α_*-inner additivity holds. Complementary additivity requires that $[0, \alpha^*] \cup [\alpha_*, \frac{1}{2}] = [0, \frac{1}{2}]$ hence $\alpha^* \geqslant \alpha_*$. If $\alpha^* > \alpha_*$, then additivity holds everywhere and expected utility holds. Otherwise, there is a unique $\kappa := \alpha^* = \alpha_*$ such that κ-inner and κ-outer additivity both hold simultaneously.

Another implication of complementary additivity is that coordinate independence holds. Let $r_i p \succsim r_i q$. Suppose that $s_i \leqslant \kappa \leqslant 1 - \kappa \leqslant r_i$. All other cases are similarly established. Given $r_i p \succsim r_i q$, $(1-\kappa)$-upper additivity implies $(1-\kappa)_i p = r_i p + (1 - \kappa - r_i)_i \succsim r_i q + (1 - \kappa - r_i)_i = (1 - \kappa)_i q$. Furthermore, $\kappa_i p \succsim \kappa_i q$ follows from κ-inner additivity, and $s_i p \succsim s_i q$ follows from κ-lower additivity. Hence, coordinate independence holds and, by Theorem 4, preferences admit a rank-dependent utility representation, with a cardinal utility u for outcomes and unique probability weighting function w that is continuous and strictly increasing.

Furthermore, because preferences satisfy κ-inner additivity and κ-outer additivity, the probability weighting function is affine on each of the intervals $[0, \kappa]$, $[\kappa, 1-\kappa]$, and $[1-\kappa, 1]$. That is, piecewise linear rank-dependent utility holds. ■

References

[1] Abdellaoui, M.: A genuine rank-dependent generalization of the von Neumann-Morgenstern expected utility theorem. Econometrica 70(2), 717-736 (2002)

[2] Abdellaoui, M. and B. Munier: On the fundamental risk-structure dependence of individual preferences under risk: An experimental investigation. Ann Oper Res 80, 237-252 (1998)

[3] Abdellaoui, M., O. L'Haridon and H. Zank: Separating curvature and elevation: A parametric probability weighting function. J Risk Uncertainty 4, 39-65 (2010)

[4] Allais, M. The foundations of a positive theory of choice involving risk and a criticism of the postulates and axioms of the American school. In Allais and Hagen (eds.), Expected utility and the Allais paradox. Dordrecht, The Netherlands: D. Reidel Publishing Company. (1952)

[5] Chateauneuf, A., J. Eichberger and S. Grant: Choice under uncertainty with best and worst in mind: NEO-additive capacities. J Econ Theory, 137, 538-567 (2007)

[6] Cohen, M.: Security level, potential level, expected utility: a three-criteria decision model under risk. Theory Dec 33, 101-134 (1992)

[7] Diecidue, E., U. Schmidt and H. Zank: Parametric weighting functions. J Econ Theory 144(3), 1102-1118 (2009)

[8] Debreu, G.: Topological methods in cardinal utility theory. In K. J. Arrow, S. Karlin, and P. Suppes (eds.): Mathematical methods in the Social Sciences, 16-26, Stanford University Press, Stanford, CA. (1960)

[9] Dominiak, A., J. Eichberger and J-P. Lefort: Agreeable trade with pessimism and optimism. Math Soc Sci 46, 119-126 (2012)

[10] Dominiak, A. and J-P. Lefort: Agreement theorem for neo-additive beliefs. Econ Th 52, 1-13 (2013)

[11] Eichberger, J., S. Grant and J-P Lefort: Generalized neo-additive capacities and updating. Int J Econ Theory 8:3, 237-257 (2012)

[12] Eichberger, J. and D. Kelsey: Are the treasures of game theory ambiguous? Econ Th 48, 313-339 (2011)

[13] Eichberger, J. and D. Kelsey: Optimism and pessimism in games, Int Econ Rev 55, 483-505 (2014)

[14] Köbberling, V. and P. P. Wakker. Preference foundations for nonexpected utility: A generalized and simplified technique. Math Oper Res 28, 395-423 (2003)

[15] Krantz, D.H., R.D. Luce, P. Suppes, and A. Tversky: Foundations of measurement, vol. I (Additive and polynomial representations). Academic Press, New York (1971)

[16] Lopes, L.L.: Between hope and fear: The psychology of risk. Advances in Experimental Psychology 20, 255-295 (1987)

[17] Lopes, L.L.: When time is of the essence: Averaging, aspiration, and the short run. Org Behav Hum Dec Proc 65, 179-189 (1996)

[18] Ludwig, A. and A. Zimper: Biased Bayesian learning with an application to the risk-free rate puzzle. J Econ Dynamics and Control 39, 79-97 (2014)

[19] Quiggin, J.: A theory of anticipated utility. J Econ Behav Org 3, 323-343 (1982)

[20] Romm, A.T.: An interpretation of focal point responses as non-additive beliefs. Judgment and Decision Making 9:5, 387-402 (2014)

[21] Schmidt, U. and A. Zimper: Security and potential preferences with thresholds. J Math Psych 51, 279-289 (2007)

[22] Teitelbaum, J.C.: A unilateral accident model under ambiguity. J Legal Stud 36, 431-477 (2007)

[23] Wakker, P.P.: Additive representations on rank-ordered sets II: The topological approach. J Math Econ 22, 1-26 (1993)

[24] Wakker, P. P.: Separating marginal utility and probabilistic risk aversion. Theory and Decision 36, 1-44 (1994)

[25] Webb, C.S. and H. Zank: Accounting for optimism and pessimism in expected utility. J Math Econ 47:6, 706-717 (2011)

[26] Webb, C.S.: Piecewise additivity for nonexpected utility. Forthcoming in Economic Theory (2015)

[27] Wu, G. and R. Gonzalez: Curvature of the probability weighting function. Manag Sci 42, 1676-1690 (1996)

[28] Zimper, A.: Asset pricing in a Lucas fruit tree economy with the best and worst in mind. J Econ Dynamics and Control 36:4, 610-628 (2012)